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On the Dynamics of finite-gap solutions in classical string theory

Nick Dorey and Benoît Vicedo

DAMTP, Centre for Mathematical Sciences, University of Cambridge
Wilberforce Road, Cambridge CB3 0WA, U.K.
E-mail: n.dorey@damtp.cam.ac.uk, b.vicedo@damtp.cam.ac.uk

Abstract: We study the dynamics of finite-gap solutions in classical string theory on $\mathbb{R} \times S^3$. Each solution is characterised by a spectral curve, Σ, of genus g and a divisor, γ, of degree g on the curve. We present a complete reconstruction of the general solution and identify the corresponding moduli-space, $\mathcal{M}_{\mathbb{R}}^{(2g)}$, as a real symplectic manifold of dimension $2g$. The dynamics of the general solution is shown to be equivalent to a specific Hamiltonian integrable system with phase-space $\mathcal{M}_{\mathbb{R}}^{(2g)}$. The resulting description resembles the free motion of a rigid string on the Jacobian torus $J(\Sigma)$. Interestingly, the canonically-normalised action variables of the integrable system are identified with certain filling fractions which play an important role in the context of the AdS/CFT correspondence.

Keywords: Integrable Field Theories, AdS-CFT Correspondence, Sigma Models.

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1. Introduction

The AdS/CFT correspondence continues to inspire progress both in gauge theory and in string theory. A particularly exciting recent development is the emergence of integrability in classical string theory on $AdS_5 \times S^5$ [1]. Specifically, the classical equations of motion of the string can be reformulated as the zero-curvature condition for a Lax connection which immediately leads to the existence of an infinite tower of conserved quantities. A similar Lax representation exists for classical string theory on a large family of backgrounds including symmetric spaces, group manifolds and various supersymmetric generalisations[1].

In concrete terms, the Lax representation allows one to obtain a very large class of explicit solutions of the classical string equations known as *finite-gap solutions*. Each of these solutions is characterised by an auxiliary Riemann surface Σ known as the spectral curve. Solutions are naturally classified by the genus g of Σ or, equivalently, by the number $K = g + 1$ of gaps or forbidden zones in the spectrum of the Lax operator. We will refer to configurations of fixed K as K-gap solutions. As we discuss below, it is plausible that arbitrary classical motions of the string can be obtained from an appropriate $K \to \infty$ limit of the K-gap solutions. Moreover, these solutions yield a description of the phase-space of the string analogous to that provided by the mode expansion in flat-space string theory [2, 4].

In this paper we will present a detailed study of the dynamics of finite-gap solutions in integrable classical string theory. Following [2] (see also [6]), we will focus on the case of bosonic string theory on $\mathbb{R} \times S^3$. At the classical level, this is a consistent truncation of the full string theory on $AdS_5 \times S^5$. In particular the solutions we study correspond to strings located at the centre of AdS_5 moving on an S^3 submanifold of S^5 (see [9] and references therein). The methods we use should apply equally to other cases where the string equations have a Lax representation including the full Metsaev-Tseytlin action [10] for $AdS_5 \times S^5$.

Our first result is an explicit reconstruction of the most general K-gap solution[2]. In particular, we identify the independent degrees of freedom of the solution and construct the corresponding moduli-space $\mathcal{M}_{\mathbb{R}}^{(2g)}$ as a real symplectic manifold of dimension $2g = 2K-2$. We find that the time evolution of the string admits a Hamiltonian description as an integrable system with this moduli space as its phase space. In fact the resulting integrable system is one of a large class of models constructed by Krichever and Phong [13]. As in other examples, the finite-dimensional phase-space takes the form of a Jacobian fibration over the moduli-space of the spectral curve Σ. At generic points, the resulting system corresponds to the free motion of an infinitely rigid, wound string on a flat torus of dimension K. A similar picture emerged for the case $K \leq 3$ in [14]. This construction naturally provides a set of action-angle variables for the string. By definition the angle variables are the flat coordinates on the Liouville torus normalised to have period 2π. Interestingly, the conjugate action variables turn out to be exactly equal to certain *filling fractions* [2, 3]

[1] Of course these spaces yield consistent backgrounds for first-quantised string theory only in very special cases such as that of $AdS_5 \times S^5$

[2] Similar solutions were obtained by a different method in [2, 11].

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which also play a key role in the context of AdS/CFT duality. Action-angle variables for pulsating string solutions were also discussed in [12]. In the rest of this introductory section we provide a brief overview of our main results. Full details of the calculations described are presented in the body of the paper.

1.1 The model

We consider a classical bosonic string moving on $\mathbb{R} \times S^3$. The string carries conserved Noether charges, Q_L and Q_R corresponding to the $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ isometry group of S^3. In a conformal, static gauge the string is effectively described by a single worldsheet field $g(\sigma,\tau) \in \mathrm{SU}(2) \simeq S^3$ obeying the closed string boundary condition $g(\sigma + 2\pi, \tau) = g(\sigma,\tau)$. Equivalently, we can describe the dynamics in terms of the $\mathrm{SU}(2)_R$ current $j = -g^{-1}dg$ which obeys the equations,

$$dj - j \wedge j = 0, \qquad d * j = 0\,. \tag{1.1}$$

Physically-allowed classical motions of the string must also obey the Virasoro constraint,

$$\frac{1}{2}\mathrm{tr} j_\pm^2 = -\kappa^2 \tag{1.2}$$

where $j_\pm$ are the components of the current j in worldsheet lightcone coordinates and κ is a constant related to the spacetime energy of the string. It will sometimes be convenient to work with a complexified version of the model, defined by the same equations, where the components of the current j take values in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. Solutions to the original problem are then obtained by applying appropriate reality conditions. In the following, we will refer to classical solutions in these two models as real solutions and complex solutions respectively.

A starting point for demonstrating the integrability of the model is the fact that the equations of motion (1.1) are equivalent to the flatness condition for a one-parameter family of conserved $\mathfrak{su}(2)$ currents,

$$J(x) = \frac{1}{1-x^2}(j - x * j). \tag{1.3}$$

Here $x \in \mathbb{C}$ is a complex spectral parameter. Using these flat currents, we can construct a monodromy matrix,

$$\Omega(x,\sigma,\tau) = P\overleftarrow{\exp}\int_{[\gamma(\sigma,\tau)]} J(x) \;\in\; \mathrm{SU}(2) \tag{1.4}$$

where $\gamma(\sigma,\tau)$ is a non-contractible loop on the string worldsheet based at the point (σ,τ). An immediate consequence of the flatness of the currents (1.3), is that $\Omega(x)$ undergoes isospectral evolution in the world-sheet coordinates. In other words the eigenvalues of the monodromy matrix are independent of σ and τ. As Ω takes values in SU(2), it is convenient to parametrise the eigenvalues as,

$$\lambda_\pm = \exp\left(\pm i p(x)\right).$$

Here $p(x)$ is a (multi-valued) function of the spectral parameter which is known as the *quasi-momentum* which yields a one-parameter family of conserved quantities on the worldsheet.

It is a remarkable fact, familiar from the study of other integrable PDEs [19, 23], that solutions to the string equations of motion (1.1) are uniquely characterised by the analytic behaviour of their monodromy matrix and its eigenvalues in the spectral parameter. In particular the two eigenvalues, $\lambda_{\pm}$, naturally correspond to the two branches of an analytic function defined on a double-cover of the complex x-plane. In general, the two sheets are joined by an arbitrary number of branch cuts $\mathcal{C}_I$. As the quasi-momentum appears in the exponent, it can have discontinuities across the branch cuts of the form,

$$p(x+\epsilon) + p(x-\epsilon) = 2\pi n_I, \quad x \in \mathcal{C}_I, \quad n_I \in \mathbb{Z}, \quad I = 1, \ldots, K. \tag{1.5}$$

More abstractly, the resulting double-cover of the x-plane defines a Riemann surface Σ, known as the spectral curve, on which dp is a meromorphic differential. The integers n_I then correspond to certain periods of the differential dp.

Solutions with a finite number of cuts are known as finite-gap solutions. More specifically we will consider K-gap solutions with K cuts where the corresponding spectral curve, Σ, has genus $g = K - 1$. Roughly speaking, the size of each cut provides a single continuous parameter of the solution. An important special case occurs when all K cuts shrink to zero size. The resulting solution describes a massless point-like string orbiting a great circle on S^3 at the speed of light. Re-introducing cuts of infinitesimal size corresponds to studying linearised oscillations around this point-like solution [2]. More precisely, the resulting solution is a linear superposition of K eigenmodes of the small fluctuation operator. The amplitude of each mode is set by the size of the corresponding cut $\mathcal{C}_I$ while the corresponding mode number is the integer n_I appearing in (1.5). In fact, allowing arbitrary K, the finite-gap solutions reproduce the full space of linear fluctuations around the point-like string [15]. For cuts of finite size, one obtains instead a family of fully non-linear solutions with the same number of parameters. This suggests that the finite-gap solutions are essentially a complete set of solutions of the equations of motion.

1.2 The moduli space of solutions

As mentioned above, finite-gap solutions are determined by the analytic properties of the corresponding monodromy matrix in the complex parameter x. In particular, the spectral curve determines the spectrum of conserved quantities for a given solution. On general grounds, for each conserved quantity, we also expect to find a canonically conjugate variable which evolves linearly in time. The extra information needed to specify a particular solution are the initial values of these variables.

In the following we will identify the holomorphic data which uniquely characterise the general finite-gap solution of (1.1) and (1.2). To simplify the problem we will focus on 'highest-weight' solutions where the Noether charges Q_L and Q_R are chosen to point in a fixed direction in the internal space,

$$Q_R = \frac{1}{2i} R\sigma_3, \qquad Q_L = \frac{1}{2i} L\sigma_3$$

where $\sigma_3 = \mathrm{diag}(1, -1)$ it the third Pauli matrix. This choice is preserved by the corresponding Cartan subgroup of the isometry group denoted $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$. The remaining generators of the isometry group rotate Q_L and Q_R and sweep out the full set of solutions.

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For fixed values of the constants L and R, the full set of data which characterise a highest-weight solution is,

$$\left\{\Sigma,\, dp,\, \gamma, \bar{\theta}\right\}. \tag{1.6}$$

Starting from an arbitrary hyperelliptic Riemann surface Σ and meromorphic differential dp, a set of necessary and sufficient conditions for the pair (Σ, dp) to correspond to a finite-gap solution was given in [2]. The new ingredients are a divisor γ of degree g on the spectral curve Σ and a complex "angle" $\bar{\theta}$ with $W = \exp(i\bar{\theta}) \in \mathbb{C}^*$. We find that any finite-gap solution gives rise to a unique data set. Conversely, given the data (1.6), we can reconstruct a unique $\mathfrak{sl}(2,\mathbb{C})$ current $j(\sigma,\tau)$ which obeys (1.1) and (1.2) and satisfies closed-string boundary conditions.[3] Explicit formulae for the resulting solution are given in (5.25). To obtain real solutions we must impose appropriate reality conditions on the data which are described in sections 4.3 and 5.5. We also discuss the reconstruction of the original world-sheet field $g(\sigma,\tau) \in \mathrm{SU}(2)$.

To make the correspondence between data and solutions described above precise, we define a *moduli space* $\mathcal{M}_{\mathbb{C}}$ of holomorphic data where each point corresponds to a distinct complex finite-gap solution. We will show that the reconstructed solution described above defines an injective map,

$$\mathcal{G} : \mathcal{M}_{\mathbb{C}} \rightarrow \mathcal{S}_{\mathbb{C}} \tag{1.7}$$

where the target, $\mathcal{S}_{\mathbb{C}}$, is the space of complex solutions.

The moduli space $\mathcal{M}_{\mathbb{C}}$ takes the form of a product $\mathcal{M}^{(2g)}_{\mathbb{C}} \times \mathbb{C}^*$ where the $\mathbb{C}^*$ factor is parametrised by $\bar{\theta}$ introduced above. The remaining factor $\mathcal{M}^{(2g)}_{\mathbb{C}}$, which we call the *reduced moduli space* is a complex manifold of dimension $2g$. We will obtain an explicit description of the reduced moduli space as a special case of a very general construction due to Krichever and Phong [13]. The resulting space is defined as a fibration,

$$J(\Sigma) \rightarrow \mathcal{M}^{(2g)}_{\mathbb{C}} \rightarrow \mathcal{L}. \tag{1.8}$$

The base $\mathcal{L}$ of the fibration is the moduli space of admissible pairs (Σ, dp) with fixed Noether charges L and R and the fibre is the Jacobian torus,

$$J(\Sigma) = \mathbb{C}^g/(2\pi\mathbb{Z}^g + 2\pi\Pi\mathbb{Z}^g),$$

where Π_{ij} is the period matrix of Σ. Following [13], the base $\mathcal{L}$ can be defined very precisely as a leaf in a certain smooth g-dimensional foliation of the universal moduli space of all Riemann surfaces of genus $g = K - 1$.

The construction of Krichever and Phong also provides a good set of holomorphic coordinates on $\mathcal{M}^{(2g)}_{\mathbb{C}}$. To define these coordinates, we introduce the standard basis of one-cycles on Σ with canonical intersections,

$$a_i \cap a_j = b_i \cap b_j = 0, \qquad a_i \cap b_j = \delta_{ij}.$$

[3] More precisely the solutions we construct satisfy all of the Virasoro constraints *except* the single condition that the total worldsheet momentum should vanish. At this stage there is also a residual redundancy corresponding to rigid translations of the spatial world-sheet coordinate σ. As discussed below, the momentum constraint is imposed at the end of the calculation.

for $i = 1, \ldots, g$. We also define the dual basis of holomorphic differentials ω_i, normalised as,

$$\int_{a_i} \omega_j = \delta_{ij}, \qquad \int_{b_i} \omega_j = \Pi_{ij}.$$

With a convenient choice of normalisation, holomorphic coordinates on the leaf $\mathcal{L}$ are given as,

$$S_i = \frac{1}{2\pi i} \frac{\sqrt{\lambda}}{4\pi} \int_{a_i} \left(x + \frac{1}{x} \right) dp \tag{1.9}$$

for $i = 1, \ldots, g$. As we discuss in Section 4.2, these integrals are essentially the filling fractions defined in [2, 3]. The divisor $\gamma = \gamma_1 \ldots \gamma_g$ provides coordinates $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_g)$ on the fibre $J(\Sigma)$ via the Abel map,

$$\boldsymbol{\theta} = 2\pi \sum_{j=1}^{g} \int_{\infty^+}^{\gamma_j} \boldsymbol{\omega} + \boldsymbol{\theta}_0 \tag{1.10}$$

where $\boldsymbol{\theta}_0 \in \mathbb{C}^g$ is a constant vector.[4] Together $\{S_i, \theta_i\}$ define a set of holomorphic coordinates on the total space of the fibration (1.8) and hence $\{S_i, \theta_i, \bar{\theta}\}$ are holomorphic coordinates on the moduli space $\mathcal{M}_{\mathbb{C}}$.

The reconstruction of explicit solutions reveals a remarkable feature which is a recurrent theme in the theory of integrable systems. The evolution of the solutions in the world-sheet coordinates can be described very simply by promoting the holomorphic data (i.e. the coordinates on $\mathcal{M}_{\mathbb{C}}$) to dynamical variables depending on σ and τ. The resulting evolution of the data corresponds to linear motion on the fibre $J(\Sigma)$ (and on $\mathbb{C}^*$) over a fixed point on the base $\mathcal{L}$. In terms of the angular coordinates θ_i introduced above, the evolution is,

$$\theta_i(\sigma, \tau) = \theta_i(0, 0) - k_i \sigma - w_i \tau \tag{1.11}$$

where the constant angular velocities are given as,

$$k_i = \frac{1}{2\pi} \int_{b_i} dp, \quad w_i = \frac{1}{2\pi} \int_{b_i} dq.$$

Here dp is the differential of the quasi-momentum introduced above. Correspondingly, dq is the differential of the *quasi-energy* which is defined in section 4 below. The b-periods k_i of dp are related to the constant mode numbers n_I appearing in (1.5) as $k_i = n_i - n_K$ for $i = 1, \ldots, g$. In contrast, the b-periods w_i of dq are non-trivial functions of the moduli S_i.

The solution also involves the following linear motion of the additional coordinate $\bar{\theta}$,

$$\bar{\theta} = \bar{\theta}_0 - k_{\bar{\theta}} \sigma - w_{\bar{\theta}} \tau\,. \tag{1.12}$$

As for the other coordinates, the constant angular velocities $k_{\bar{\theta}}$ and $\omega_{\bar{\theta}}$ are expressed as periods of the differentials dp and dq respectively,

$$k_{\bar{\theta}} = -\frac{1}{2\pi} \int_{\mathcal{B}_{g+1}} dp, \qquad w_{\bar{\theta}} = -\frac{1}{2\pi} \int_{\mathcal{B}_{g+1}} dq$$

where the one-cycle $\mathcal{B}_{g+1}$ is defined in the text and $k_{\bar{\theta}}$ is equal to the mode number n_K.

[4]The precise value of $\boldsymbol{\theta}_0$ is fixed in section 5.5 (see eqn (5.43)) to ensure that the coordinates θ_i real when the approriate reality conditions are imposed on the corresponding divisor γ.

With an appropriate choice for the constant $\boldsymbol{\theta}_0$ appearing in (1.10), we find that the condition for real solutions simply corresponds to restricting the coordinates $\{S_i, \theta_i, \bar{\theta}\}$ on $\mathcal{M}_{\mathbb{C}}$ to real values. We denote the corresponding real slice of the moduli space as $\mathcal{M}_{\mathbb{R}} = \mathcal{M}_{\mathbb{R}}^{(2g)} \times S^1$. In particular, the resulting real coordinates θ_i and $\bar{\theta}$ are angular variables with period 2π. The angle $\bar{\theta}$ is a coordinate describing the orientation of the string in the unbroken global symmetry group $U(1)_R$, while the remaining angles correspond to internal degrees of freedom of the string. Together, the variables θ_i and $\bar{\theta}$ parametrise a real torus $T^{g+1} \subset J(\Sigma) \times \mathbb{C}^*$ and the linear evolution (1.11) (1.12) can be understood as the free motion of a rigid string on this torus. The string is wound around a cycle on T^{g+1} which is determined by the mode numbers n_I and moves on the torus with constant angular velocity.

1.3 The integrable system

So far we have discussed the reduced moduli space $\mathcal{M}_{\mathbb{C}}^{(2g)}$ as a complex manifold without additional structure. However, the construction of Krichever and Phong also provides a globally-defined holomorphic, non-degenerate closed two-form with which $\mathcal{M}_{\mathbb{C}}^{(2g)}$ becomes a complex symplectic manifold. Similarly the real slice $\mathcal{M}_{\mathbb{R}}^{(2g)}$ becomes a real, symplectic manifold. In the coordinates $\{S_i, \theta_i\}$ introduced above, the real symplectic form is,

$$\omega_{2g} = \sum_{i=1}^{g} \delta S_i \wedge \delta\theta_i \,. \tag{1.13}$$

Here δ denotes the exterior derivative on the moduli space. Together $\mathcal{M}_{\mathbb{R}}^{(2g)}$ and ω_{2g} define a Hamiltonian integrable system having g Poisson-commuting conserved quantities $\{S_i\}$. The conjugate angle variables $\{\theta_i\}$ evolve linearly under the flows generated by these Hamiltonians. We will now show that this integrable system provides an effective Hamiltonian description of the dynamics of the string.

The gauge-fixed string action implies a non-trivial set of Poisson brackets for the current components $j_0(\sigma,\tau)$ and $j_1(\sigma,\tau)$ (see equations (2.10) below). Using these brackets, translations of the world-sheet coordinates σ and τ are generated by the momentum and energy of the 2d principal chiral model defined as

$$\begin{aligned} \mathcal{P} &= -\frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \, \mathrm{tr}\,[j_0 j_1] \\ \mathcal{E} &= -\frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \, \frac{1}{2}\mathrm{tr}\,\left[j_0^2 + j_1^2\right] \end{aligned}$$

respectively.[5] As we have already accounted for all the moduli of the solution, we should think of $\mathcal{P}$ and $\mathcal{E}$ as functions on the space of solutions. Further, as these quantities are manifestly conserved, they correspond to well-defined functions on the base $\mathcal{L}$ of the

[5]Ultimately, the Virasoro constraints (1.2) imply that $\mathcal{P} = 0$. As mentioned above, we will delay imposing this final constraint until the end of the calculation.

fibration (1.8). We now define corresponding Hamiltonian functions,

$$\begin{aligned} H_\sigma &= \mathcal{P}[S_1, \ldots, S_g] \\ H_\tau &= \mathcal{E}[S_1, \ldots, S_g] \end{aligned} \tag{1.14}$$

for the integrable system defined by the manifold $\mathcal{M}_{\mathbb{R}}^{(2g)}$ equipped with the symplectic form ω_{2g} given in (1.13) above. Our main result is that the linear flow on $\mathcal{M}_{\mathbb{R}}^{(2g)}$ generated by H_σ and H_τ, with parameters σ and τ is precisely the same as the linear evolution (1.11) defined by the finite-gap solution.

It is striking that the Hamiltonians (1.14) reproduce the correct evolution precisely when they act on the holomorphic data with the Poisson brackets defined by the natural symplectic form ω_{2g} defined in [13]. Of course, imposing the correct evolution under two Hamiltonians does not uniquely fix the symplectic form and it would be preferable to derive the symplectic structure on the moduli space of finite-gap solutions directly from the string action. In a forthcoming paper [17] we will show that the symplectic form ω_{2g} on $\mathcal{M}_{\mathbb{R}}^{(2g)}$ is indeed a consequence of the Poisson brackets implied by the string action. The latter define a symplectic form ω_∞ on the full infinite-dimensional phase space of the string. The form ω_{2g} then arises naturally when ω_∞ is restricted to the space of K-gap solutions. Similar results exist for several other integrable non-linear PDEs including the equations of the KP and KdV hierarchies and the Toda equations [19, 22, 13]. The present example is more complicated as the infinite-dimensional symplectic structure is non-ultra local, leading to ambiguities in the brackets of the monodromy matrices which require regularisation. This problem was addressed in a series of papers by Maillet [16]. In [17] we will show that the symmetric regularisation prescription advocated in these papers, when applied to the present model, leads directly to the symplectic form defined in (1.13) above.

Another striking feature is that the correctly-normalised[6] action variables of the integrable system are the filling fractions defined in (1.9) above. In a leading-order semiclassical quantisation of the integrable system, the Bohr-Sommerfeld condition dictates that these action variables are quantised in integer units. As mentioned above, the K-gap solutions discussed here correspond to motions of a classical string on an $\mathbb{R} \times S^3$ submanifold of $AdS_5 \times S^5$. A leading-order semiclassical quantisation of these solutions, valid for $\lambda \gg 1$, would also involve applying the Bohr-Sommerfeld quantisation condition on periodic orbits of the string [18]. This should coincide with the semiclassical quantisation of the integrable system described above[7] and therefore lead to a discrete spectrum of string states where the filling fractions are quantised in integer units. This would be very natural in the context of the AdS/CFT correspondence where the filling fractions correspond to the number of Bethe roots of each mode number in the dual spin chain description.

As it stands the Hamiltonian description of the dynamics of the data given above is not quite complete. In particular, we have only discussed the (σ, τ)-evolution of the data

[6]By this we mean that the canonically conjugate angle variables are normalised to have period 2π.

[7]Note that the familiar sum over the zero point energies of small fluctuations around a classical solution appears at the *next* order in the semiclassical expansion. Obviously this includes fluctuations of the target space coordinates outside the $\mathbb{R} \times S^3$ submanifold and would not be captured by a quantisation of the integrable system proposed here.

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on the reduced moduli space $\mathcal{M}_{\mathbb{R}}^{(2g)}$ and not on the full moduli space $\mathcal{M}_{\mathbb{R}}$. The difference is the additional S^1 parametrised by the angle $\bar{\theta}$ whose linear evolution is given in (1.12). As mentioned above, the evolution of $\bar{\theta}$ corresponds to a global $U(1)_R$ rotation of the string with constant angular velocity. As in the study of rotating rigid bodies, we can remove this motion by working in an appropriate reference frame which rotates with the string. As we discuss in subsection 5.8, the motion on the reduced moduli space $\mathcal{M}_{\mathbb{R}}^{(2g)}$ naturally corresponds to the motion of the string in this rotating frame.

2. The σ-model on $\mathbb{R} \times S^3$

The action for a string on $\mathbb{R} \times S^3$ in conformal gauge reads

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \left[\sum_i \partial_a X_i \partial^a X_i - \partial_a X_0 \partial^a X_0 + \Lambda \left(\sum_i X_i^2 - 1 \right) \right]$$

where X_0 is the time coordinate, X_i are Cartesian coordinates on $\mathbb{R}^4$ and Λ is a Lagrange multiplier constraining the string to the unit sphere $S^3 \subset \mathbb{R}^4$. In terms of worldsheet lightcone coordinates

$$\sigma^\pm = \frac{1}{2}(\tau \pm \sigma) = \frac{1}{2}(\sigma^0 \pm \sigma^1), \quad \partial_\pm = \partial_0 \pm \partial_1, \tag{2.1}$$

the equations of motion derived from this action read

$$\partial_+ \partial_- X_i + \left(\sum_j \partial_+ X_j \partial_- X_j \right) X_i = 0, \quad \partial_+ \partial_- X_0 = 0,$$

which must be supplemented by the Virasoro constraints

$$\sum_i (\partial_\pm X_i)^2 = (\partial_\pm X_0)^2.$$

The equation for X_0 being decoupled from the other fields it can be solved separately and has the general solution $X_0 = \kappa\tau + f_+(\sigma^+) + f_-(\sigma^-)$. Using the residual gauge symmetry $\sigma^\pm \to F_\pm(\sigma^\pm)$ one can gauge away $f_\pm$, leaving $X_0 = \kappa\tau$. The Virasoro constraint becomes

$$\sum_i (\partial_\pm X_i)^2 = \kappa^2.$$

In this gauge the spacetime energy of the string is given as,

$$\Delta = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma\, \partial_\tau X_0 = \sqrt{\lambda}\kappa$$

and we therefore identify the constant κ as $\Delta/\sqrt{\lambda}$.

2.1 SU(2) principal chiral model

Since the sphere S^3 is the group SU(2), the σ-model in question can be reformulated as an SU(2) principal chiral model for a field g taking values in SU(2) by defining

$$g = \begin{pmatrix} X_1 + iX_2 & X_3 + iX_4 \\ -X_3 + iX_4 & X_1 - iX_2 \end{pmatrix} \in \mathrm{SU}(2). \tag{2.2}$$

In terms of the current[8] $j = -g^{-1}dg$, the action may be rewritten as follows

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int \left[\frac{1}{2}\mathrm{tr}(j \wedge *j) + dX_0 \wedge *dX_0 \right]. \tag{2.3}$$

The current j is identically flat from its definition and the equations of motion are equivalent to its conservation

$$dj - j \wedge j = 0, \quad d * j = 0. \tag{2.4}$$

In static gauge, $X_0 = \kappa\tau$, the conserved Noether charges associated with translations of the world-sheet coordinates σ and τ are $\mathcal{P}$ and $\mathcal{E} - \sqrt{\lambda}\kappa^2/2$ where,

$$\begin{aligned} \mathcal{P} &= -\frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \,\mathrm{tr}\left[j_0 j_1\right] \\ \mathcal{E} &= -\frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \,\frac{1}{2}\mathrm{tr}\left[j_0^2 + j_1^2\right] \end{aligned} \tag{2.5}$$

are the two-dimensional momentum and energy of the principal chiral model respectively.

In terms of the lightcone components $j_\pm = j_0 \pm j_1$ of the current j, the Virasoro constraints read,

$$\frac{1}{2}\mathrm{tr} j_\pm^2 = -\kappa^2. \tag{2.6}$$

For later convenience, it will be useful to split these constraints into two parts. The first set of constraints read,

$$\frac{1}{2}\mathrm{tr} j_\pm^2 = -\kappa_\pm^2. \tag{2.7}$$

where, for the moment, $\kappa_\pm$ are undetermined constants. The world-sheet momentum and energy defined above then become,

$$\begin{aligned} \mathcal{P} &= \frac{\sqrt{\lambda}}{4}(\kappa_+^2 - \kappa_-^2), \\ \mathcal{E} &= \frac{\sqrt{\lambda}}{4}(\kappa_+^2 + \kappa_-^2). \end{aligned} \tag{2.8}$$

The remaining content of the Virasoro constraint (2.6) is the vanishing of the total momentum $\mathcal{P} = 0$ which implies $\kappa_+^2 = \kappa_-^2 = \kappa^2$ and the string mass-shell condition,

$$\mathcal{E} = \frac{\sqrt{\lambda}}{2}\kappa^2 = \frac{\Delta^2}{2\sqrt{\lambda}}. \tag{2.9}$$

[8] Note the sign difference in the definition of j with [2]. The choice of sign here is to agree with the literature on algebro-geometric methods and finite-gap integration [19 – 21, 23].

It will also be useful to consider the Hamiltonian description of string dynamics in static gauge. The string action (2.3) leads to a set of non-trivial equal-τ Poisson brackets for the current components, $j_0(\sigma)$ and $j_1(\sigma)$. Writing the currents as $j_0 = j_0^a t^a$, $j_1 = j_1^a t^a$ in terms of SU(2) generators t^a satisfying,

$$[t^a, t^b] = f^{abc} t^c, \quad \mathrm{tr}(t^a t^b) = -\delta^{ab}.$$

the Poisson brackets are given as,

$$\begin{aligned} \left\{ j_1^a(\sigma), j_1^b(\sigma') \right\} &= 0, \\ \frac{\sqrt{\lambda}}{4\pi} \left\{ j_0^a(\sigma), j_1^b(\sigma') \right\} &= -f^{abc} j_1^c(\sigma) \delta(\sigma - \sigma') - \delta^{ab} \delta'(\sigma - \sigma'), \\ \frac{\sqrt{\lambda}}{4\pi} \left\{ j_0^a(\sigma), j_0^b(\sigma') \right\} &= -f^{abc} j_0^c(\sigma) \delta(\sigma - \sigma'). \end{aligned} \tag{2.10}$$

The resulting symplectic structure is non-ultra-local reflecting the presence of the term proportional to $\delta'(\sigma - \sigma')$ in the second bracket. This term leads to some ambiguities in the Poisson brackets of the monodromy matrix of the model. However, the brackets lead (unambiguously) to a Hamiltonian version of the equations of motion (2.4) which read,

$$\partial_\sigma j = \{\mathcal{P}, j\}, \quad \partial_\tau j = \{\mathcal{E}, j\} \tag{2.11}$$

where $\mathcal{P}$ and $\mathcal{E}$ are the world-sheet momentum and energy defined above. We will not make any further use of these Poisson brackets in this paper other than to note the fact that the σ and τ evolution of the current j are generated by $\mathcal{P}$ and $\mathcal{E}$ respectively. A more complete discussion of the Hamiltonian formalism for this model including the issues raised by non-ultra-locality will appear in a forthcoming paper [17].

2.2 Symmetries

The action (2.3) has a global $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry

$$g \to U_L g U_R,$$

where U_L and U_R are constant matrices. The Noether current corresponding to $\mathrm{SU}(2)_R$ is the current $j = -g^{-1}dg$ introduced above whereas the Noether current for the $\mathrm{SU}(2)_L$ symmetry is $l = -dg\, g^{-1} = gjg^{-1}$. The corresponding Noether charges are

$$\begin{aligned} \mathrm{SU}(2)_R: \quad Q_R &= \frac{\sqrt{\lambda}}{4\pi} \int_\gamma *j = -\frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma j_0, \\ \mathrm{SU}(2)_L: \quad Q_L &= \frac{\sqrt{\lambda}}{4\pi} \int_\gamma *l = -\frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma l_0, \end{aligned}$$

where γ is any curve winding once around the world-sheet, expressing the conservation of these Noether charges (see figure 1), e.g.

$$\int_{\gamma_2} *j - \int_{\gamma_1} *j = \int_{\partial D} *j = \int_D d * j = 0.$$

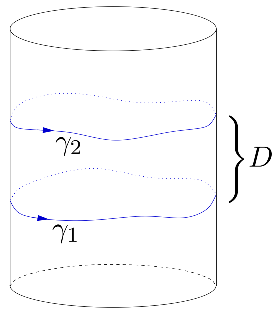

Figure 1: Conservation of Q_R and Q_L.

Notice that the $\mathrm{SU}(2)_R$ current j which appears in the action (2.3) is invariant under the action of $\mathrm{SU}(2)_L$. On the other hand the $\mathrm{SU}(2)_R$ symmetry acts non-trivially on the current as

$$j \to U_R^{-1} j U_R. \tag{2.12}$$

3. Classical integrability

The equations (2.4) of the SU(2) principal chiral model can be reformulated as a 1-parameter family of zero-curvature equations which is one of the most general representations known to give rise to classical integrable systems and constitutes the starting point for constructing solutions by algebro-geometric methods [19 – 21, 23].

3.1 Lax connection and monodromy

We search for the 1-parameter family of flat currents in the form $J(x) = \alpha(x) j + \beta(x) * j$. Using (2.4) we find $dJ(x) - J(x) \wedge J(x) = -(\alpha(x)^2 - \alpha(x) - \beta(x)^2) j \wedge j$ so that $J(x)$ is flat provided $\alpha(x)^2 - \alpha(x) - \beta(x)^2 = 0$ which is solved by $\alpha(x) = \frac{1}{1-x^2}$, $\beta(x) = \frac{-x}{1-x^2}$. Therefore

$$J(x) = \frac{1}{1-x^2}(j - x * j), \tag{3.1}$$

or in components $J(x) = J_0 d\tau + J_1 d\sigma$ (with the convention $\epsilon_{01} = 1$, $(*j)_\mu = \epsilon_{\nu\mu} j^\nu$)

$$\begin{aligned} J_0(x, \sigma, \tau) &= \frac{1}{2}\left(\frac{j_+}{1-x} + \frac{j_-}{1+x}\right) \\ J_1(x, \sigma, \tau) &= \frac{1}{2}\left(\frac{j_+}{1-x} - \frac{j_-}{1+x}\right). \end{aligned}$$

By construction, the equation

$$dJ(x, \sigma, \tau) - J(x, \sigma, \tau) \wedge J(x, \sigma, \tau) = 0 \tag{3.2}$$

holds identically in x and is equivalent to the pair of equations in (2.4). Given such a family of flat connections $J(x, \sigma, \tau)$ on the world-sheet it is natural to consider the corresponding parallel transporters along any given curve γ. By virtue of the flatness condition (3.2) the

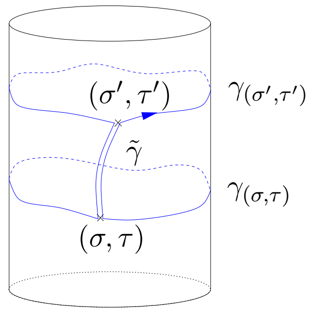

Figure 2: (σ, τ)-evolution of the monodromy matrix $\Omega(x, \sigma, \tau)$.

non-Abelian version of Stokes's theorem implies that for any simply connected domain D (so that ∂D is a single closed curve) one has

$$P\overleftarrow{\exp}\left\{\int_{\partial D} J(x, \sigma, \tau)\right\} = \mathbf{1}. \tag{3.3}$$

It follows that the parallel transporter along any given curve γ will only depend on the homotopy class $[\gamma]$ of γ with fixed end points. Of particular interest is the monodromy matrix defined as the parallel transporter along a closed loop $\gamma(\sigma, \tau)$ bound at (σ, τ) and winding once around the world-sheet

$$\Omega(x, \sigma, \tau) = P\overleftarrow{\exp}\int_{[\gamma(\sigma,\tau)]} J(x, \sigma, \tau) = P\overleftarrow{\exp}\int_{\sigma}^{\sigma+2\pi} d\sigma' \frac{1}{2}\left(\frac{j_+(\sigma', \tau)}{1-x} - \frac{j_-(\sigma', \tau)}{1+x}\right). \tag{3.4}$$

Equation (3.3) implies that monodromy matrices based at different points (σ, τ) and (σ', τ') are related by conjugation (see Figure 2)

$$\Omega(x, \sigma', \tau') = U\Omega(x, \sigma, \tau)U^{-1}, \quad \text{with } U = P\overleftarrow{\exp}\int_{\tilde{\gamma}} J(x, \sigma, \tau), \tag{3.5}$$

where $\tilde{\gamma}$ is any curve connecting the two points (σ, τ) and (σ', τ') in question.

3.2 Analyticity

Since the dependence of $\Omega(x, \sigma, \tau)$ on (σ, τ) is governed by

$$[d - J(x, \sigma, \tau), \Omega(x, \sigma, \tau)] = 0, \tag{3.6}$$

it follows from Poincaré's theorem on holomorphic differential equations that $\Omega(x, \sigma, \tau)$ is holomorphic in x except at the singular points $x = \pm 1$ of the Lax connection $J(x, \sigma, \tau)$ where it has essential singularities. Also, since $\mathrm{tr} j_\pm = 0$ it follows that the monodromy matrix is unimodular, i.e. $\det \Omega(x, \sigma, \tau) = 1$. So at generic values of x where it can be diagonalised it takes on the following diagonal form

$$u(x, \sigma, \tau)\Omega(x, \sigma, \tau)u(x, \sigma, \tau)^{-1} = \mathrm{diag}\left(e^{ip(x)}, e^{-ip(x)}\right), \tag{3.7}$$

where the *quasi-momentum* $p(x)$ has simple poles at $x = \pm 1$. But even though $\Omega(x, \sigma, \tau)$ is meromorphic in $x \in \mathbb{CP}^1 \setminus \{x = \pm 1\}$, the quasi-momentum $p(x)$ is only well defined on the x-plane with branch cuts [3, 4] since going around a simple zero of $\Delta(x) = \left(e^{ip(x)} - e^{-ip(x)}\right)^2$ is easily seen to interchange $e^{ip(x)}$ with $e^{-ip(x)}$.

The asymptotic behaviour of the monodromy matrix near $x = \pm 1$ follows directly from (3.4)

$$\Omega(x, \sigma, \tau) = P\overleftarrow{\exp} \int_{\sigma}^{\sigma+2\pi} d\sigma' \left[-\frac{1}{2} \frac{j_{\pm}(\sigma', \tau)}{x \mp 1} + O\left((x \mp 1)^0\right) \right], \quad \text{as } x \to \pm 1$$

It is diagonalised to leading order in $(x \mp 1)$ by the eigenvectors of $j_{\pm}$ using the relation $u(\sigma, \tau) j_{\pm}(\sigma, \tau) u(\sigma, \tau)^{-1} = i\kappa\sigma_3$, where $\sigma_3 = \mathrm{diag}(1, -1)$ is the third Pauli matrix. Sub-leading orders are diagonalised by adding to $u(\sigma, \tau)$ higher orders in $(x \mp 1)$ to obtain a matrix $u(x, \sigma, \tau) = u(\sigma, \tau) + O(x \mp 1)$ analytic in x such that

$$u(x, \sigma, \tau)\Omega(x, \sigma, \tau)u(x, \sigma, \tau)^{-1} = \exp\left[-\frac{i\pi\kappa}{x \mp 1}\sigma_3 + O\left((x \mp 1)^0\right) \right], \quad \text{as } x \to \pm 1.$$

In the definition (3.7) of the quasi-momentum $p(x)$ there is an ambiguity coming from the freedom of swapping the two sheets. We fix this ambiguity by choosing the asymptotic behaviour of $p(x)$ at $x = \pm 1$ to be

$$p(x) = -\frac{\pi\kappa}{x \mp 1} + O\left((x \mp 1)^0\right), \quad \text{as } x \to \pm 1. \tag{3.8}$$

The other possible choice for the asymptotics of the quasi-momentum near $x = \pm 1$ would be to take opposite relative signs for the pole terms at $x = 1$ and $x = -1$, namely to replace $p(x)$ by the function $q(x)$ with the following asymptotics

$$q(x) = \mp\frac{\pi\kappa}{x \mp 1} + O\left((x \mp 1)^0\right), \quad \text{as } x \to \pm 1. \tag{3.9}$$

This function will play an important role in the analysis with (3.8) as quasi-momentum and so we will refer to it as the *quasi-energy*.

3.3 Asymptotics

The asymptotic expansion of the connection (3.1) at $x = \infty$

$$J(x) = \frac{1}{x} * j + O\left(\frac{1}{x^2}\right), \tag{3.10}$$

leads to the following asymptotic expansion of the monodromy matrix at $x = \infty$

$$\begin{aligned}
\Omega(x, \sigma, \tau) &= P\overleftarrow{\exp} \int_{[\gamma(\sigma,\tau)]} \left(\frac{1}{x} * j + O\left(\frac{1}{x^2}\right) \right) \\
&= \mathbf{1} - \frac{1}{x}\int_{\sigma}^{\sigma+2\pi} d\sigma' j_0 + O\left(\frac{1}{x^2}\right), \quad \text{as } x \to \infty \\
&= \mathbf{1} + \frac{1}{x}\frac{4\pi Q_R}{\sqrt{\lambda}} + O\left(\frac{1}{x^2}\right), \quad \text{as } x \to \infty.
\end{aligned} \tag{3.11}$$

Since the Noether charges Q_R and Q_L are conserved classically, we may fix them to lie in a particular direction of $\mathfrak{su}(2)$ and take them for example to be proportional to the third Pauli matrix $\sigma_3 = \mathrm{diag}(1, -1)$

$$Q_R = \frac{1}{2i} R\sigma_3, \; Q_L = \frac{1}{2i} L\sigma_3, \quad R, L \in \mathbb{R}_+.$$

where R and L are constants of the motion. By restricting the Noether charges in this way we will only obtain the restricted set of 'highest weight' solutions to the equations of motion, but all other solutions can be obtained from these by applying a combination of $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_L$ to such a 'highest weight' solution. With this restriction the asymptotic expansion of the monodromy matrix at $x = \infty$ reduces to

$$\Omega(x, \sigma, \tau) = \mathbf{1} - \frac{1}{x}\frac{2\pi i R}{\sqrt{\lambda}}\sigma_3 + O\left(\frac{1}{x^2}\right), \quad \text{as } x \to \infty. \tag{3.12}$$

We have the freedom of choosing the branch of the logarithm in the definition of the quasi-momentum and we do so such that $p(x) \sim O(1/x)$ as $x \to \infty$, in other words

$$p(x) = -\frac{1}{x}\frac{2\pi R}{\sqrt{\lambda}} + O\left(\frac{1}{x^2}\right), \quad \text{as } x \to \infty. \tag{3.13}$$

The asymptotics of the connection at $x = 0$ is $J(x) = j - x * j + O\left(x^2\right)$, so that

$$\begin{aligned} d - J(x) &= d - j + x * j + O\left(x^2\right), \\ &= g^{-1}\left(d + x * l + O\left(x^2\right)\right) g, \end{aligned}$$

where $l = -dg\, g^{-1} = gjg^{-1}$. Now because the field $g(\sigma, \tau)$ is periodic in σ it follows that the asymptotic expansion of the monodromy matrix near $x = 0$ is given by

$$\begin{aligned} g(\sigma,\tau)\Omega(x,\sigma,\tau)g^{-1}(\sigma,\tau) &= P\overleftarrow{\exp}\left(\int_{[\gamma(\sigma,\tau)]} -x * l + O\left(x^2\right)\right), \\ &= \mathbf{1} + x\int_{\sigma}^{\sigma+2\pi} d\sigma' l_0 + O\left(x^2\right), \quad \text{as } x \to 0 \\ &= \mathbf{1} - x\frac{4\pi Q_L}{\sqrt{\lambda}} + O\left(x^2\right), \quad \text{as } x \to 0. \\ &= \mathbf{1} + x\frac{2\pi i L}{\sqrt{\lambda}}\sigma_3 + O\left(x^2\right), \quad \text{as } x \to 0, \end{aligned} \tag{3.14}$$

which implies the following asymptotic expansion of the quasi-momentum at $x = 0$

$$p(x) = 2\pi m + x\frac{2\pi L}{\sqrt{\lambda}} + O\left(x^2\right), \quad \text{as } x \to 0, \tag{3.15}$$

where $m \in \mathbb{Z}$. The integer m corresponds to the winding number of the string around an equator of S^3. Although $\pi_1(S^3) = 0$, a non-trivial winding number emerges when we restrict our attention to highest weight solutions as described in the previous subsection. For convenience we will focus on the sector with $m = 0$, which includes the pointlike string solution. Our results can easily be generalised to the other sectors.

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3.4 Reality conditions

The monodromy matrix also satisfies a reality condition coming from the property $j_\pm^\dagger = -j_\pm$ of the matrices $j_\pm \in \mathfrak{su}(2)$, namely

$$\Omega(x,\sigma,\tau)^\dagger = \Omega(\bar{x},\sigma,\tau)^{-1}, \tag{3.16}$$

so in particular, for $x \in \mathbb{R}$ we have $\Omega(x,\sigma,\tau) \in \mathrm{SU}(2)$. Since the eigenvalues of $\Omega(x,\sigma,\tau)$ are $\{e^{ip(x)}, e^{-ip(x)}\}$ it follows that for $x \in \mathbb{R}$ in the cut plane, where $p(x)$ is well defined, we have $p(x) \in \mathbb{R}$. Using (3.16) the eigenvalues $\{e^{ip(\bar{x})}, e^{-ip(\bar{x})}\}$ of $\Omega(\bar{x},\sigma,\tau)$ can be also written as $\{e^{i\overline{p(x)}}, e^{-i\overline{p(x)}}\}$, but by the previous argument $p(\bar{x}) = \overline{p(x)}$ when $x \in \mathbb{R}$ in the cut plane and so by continuity $e^{ip(\bar{x})} = e^{i\overline{p(x)}}$ for all x in the cut plane. In other words $p(\bar{x}) = \overline{p(x)} + 2\pi k,\ k \in \mathbb{Z}$, but then $p(x) = \overline{p(\bar{x})} + 2\pi k = p(x) + 4\pi k$ so that $k = 0$. Hence for all x where $p(x)$ is defined it satisfies the reality condition

$$\overline{p(x)} = p(\bar{x}). \tag{3.17}$$

Thus, in particular, $p(x)$ is real on the real axis.

3.5 Gauge redundancy

The zero-curvature equation (3.2) is invariant under gauge transformations

$$J(x,\sigma,\tau) \to \tilde{g} J(x,\sigma,\tau) \tilde{g}^{-1} + d\tilde{g}\tilde{g}^{-1}, \tag{3.18}$$

where $\tilde{g} = \tilde{g}(\sigma,\tau)$ is periodic in σ and under which $\Omega(x,\sigma,\tau) \to \tilde{g}\Omega(x,\sigma,\tau)\tilde{g}^{-1}$. The gauge transformation parameter $\tilde{g}$ must be independent of x so as to preserve the analytical properties in x of the connection $J(x,\sigma,\tau)$ and of $\Omega(x,\sigma,\tau)$ (and thus of the spectral curve, introduced in section 4). Preserving also the reality condition (3.16) on $\Omega(x,\sigma,\tau)$ coming from $j_\pm^\dagger = -j_\pm$ leads to the further restriction $\tilde{g}(\sigma,\tau) \in \mathrm{SU}(2)$. By equation (3.12) the matrix of eigenvectors $u^{-1}(x,\sigma,\tau)$ of $\Omega(x,\sigma,\tau)$ introduced in (3.7) is diagonal above $x = \infty$,

$$u^{-1}(x,\sigma,\tau) = \mathbf{1} + O\left(\frac{1}{x}\right) \quad \text{as } x \to \infty,$$

and in order to preserve this condition one has to further restrict $\tilde{g}(\sigma,\tau) \in \mathrm{SU}(2)$ to be diagonal. In other words, the level set $Q_R = \frac{1}{2i} R\sigma_3$ is preserved only by a residual $U(1)_R$ action of the full $\mathrm{SU}(2)_R$ gauge group.

4. The spectral curve

Monodromy matrices at different points (σ,τ) are related by similarity transformations (3.5) and so the eigenvalues $\{e^{ip(x)}, e^{-ip(x)}\}$ of $\Omega(x,\sigma,\tau)$ are independent of (σ,τ). This motivates the definition of an invariant curve in $\mathbb{C}^2$ as the characteristic equation of $\Omega(x,\sigma,\tau)$, namely

$$\Gamma: \quad \Gamma(x,y) = \det(y\mathbf{1} - \Omega(x,\sigma,\tau)) = 0, \tag{4.1}$$

which is independent of (σ,τ). In the present case Γ is represented as a 2-sheeted ramified cover of the x-plane and as such it has a natural hyperelliptic holomorphic involution which exchanges the two sheets

$$\hat{\sigma}: \Gamma \to \Gamma, (x,y) \mapsto (x, y^{-1}). \tag{4.2}$$

4.1 The algebraic curve

The problem with the curve in (4.1) is that it doesn't define an algebraic curve in $\mathbb{C}^2$ since the eigenvalues $\{e^{ip(x)}, e^{-ip(x)}\}$ have essential singularities at $x = \pm 1$; so in particular the curve Γ has infinitely many singular points $\{x_k\}$ at which the two sheets meet $e^{ip(x_k)} = e^{-ip(x_k)}$ and which accumulate at $x = \pm 1$ (since the function $e^{2ip(x)}$ must take the value 1 infinitely many times in any neighbourhood of $x = \pm 1$). This can however be remedied [3] by defining a new matrix $L(x, \sigma, \tau)$ from $\Omega(x, \sigma, \tau)$

$$u(x,\sigma,\tau)L(x,\sigma,\tau)u(x,\sigma,\tau)^{-1} = -i\frac{\partial}{\partial x}\log\left(u(x,\sigma,\tau)\Omega(x,\sigma,\tau)u(x,\sigma,\tau)^{-1}\right),$$

which has the same eigenvectors as $\Omega(x, \sigma, \tau)$ but the corresponding eigenvalues $\{p'(x), -p'(x)\}$ are meromorphic in x with only double poles at $x = \pm 1$. Using this new matrix $L(x, \sigma, \tau)$ we can now define an algebraic curve in $\mathbb{C}^2$ by

$$\widehat{\Sigma}: \quad \widehat{\Sigma}(x, y) = \det(y\mathbf{1} - L(x, \sigma, \tau)) = 0. \tag{4.3}$$

A complete discussion of the properties of this algebraic curve for strings on $\mathbb{R} \times S^p$ can be found in [3] and those of the full algebraic curve for superstrings on $AdS_5 \times S^5$ in [4]; see also [5, 7, 8]. We show in Appendix A that by performing a birational transformation on the variable y to remove the (finitely many) singular points it can be turned into the following hyperelliptic form

$$\Sigma : y^2 = \prod_{I=1}^{K}(x - u_I)(x - v_I), \tag{4.4}$$

where K is the number of branch cuts $\mathcal{C}_I$ which connect the branch points $u_I, v_I, I = 1, \dots, K$. A solution for which the corresponding curve has finitely many branch points and hence finite genus $g = K - 1$ is called a *finite-gap solution.* Let $\hat{\pi} : \Sigma \to \mathbb{CP}^1$ denote the hyperelliptic projection of Σ onto the x-plane. We shall refer to the sheet of this double cover corresponding to $p(x)$ as the physical sheet, and for any value $x \in \mathbb{CP}^1$ we shall denote the pair of points in $\hat{\pi}^{-1}(x)$ as $x^{\pm}$ (related to each other by $x^+ = \hat{\sigma}(x^-)$), with x^+ being the point on the physical sheet.

For each cut $\mathcal{C}_I$ we define a cycle $\mathcal{A}_I$ encircling the cut on the physical sheet and a path $\mathcal{B}_I$ joining the points $x = \infty$ on both sheets through the cut $\mathcal{C}_I$ as shown in figure 3. Let us also define a basis $\{a_i, b_i\}_{i=1}^g$ for the first homology group $H_1(\Sigma, \mathbb{R})$ of the curve Σ by setting for example $a_i = \mathcal{A}_i$ and $b_i = \mathcal{B}_i - \mathcal{B}_{g+1}$. This basis has the following canonical intersection matrix

$$a_i \cap a_j = b_i \cap b_j = 0, \ a_i \cap b_j = \delta_{ij}.$$

Define also the dual basis $\{\omega_i\}_{i=1}^g$ of holomorphic differentials on Σ normalised such that $\int_{a_i} \omega_j = \delta_{ij}$. The matrix of b-periods of these holomorphic differentials defines a symmetric matrix with positive imaginary part called the period matrix Π of the Riemann surface Σ

$$\Pi_{ij} = \int_{b_i} \omega_j. \tag{4.5}$$

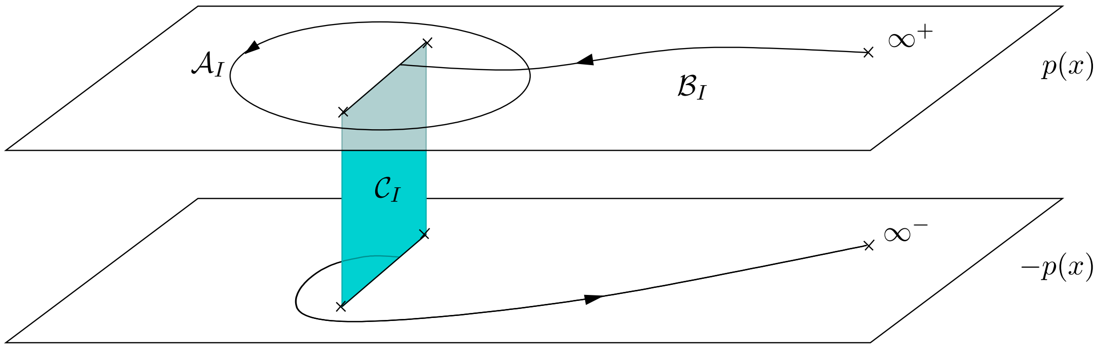

Figure 3: The cycle $\mathcal{A}_I$ and path $\mathcal{B}_I$ for the cut $\mathcal{C}_I$.

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Using this data of Σ one can define an important object associated with any Riemann surface known as the Jacobian variety and defined as the following g-dimensional complex torus

$$J(\Sigma) = \mathbb{C}^g/(2\pi\mathbb{Z}^g + 2\pi\Pi\mathbb{Z}^g).$$

Now since by definition $p(x)$ is single valued on the cut x-plane it follows that the $\mathcal{A}$-periods of dp must vanish, and using $\Omega(x) \sim_{x\to\infty} \mathbf{1}$ from (3.11) we find that the $\mathcal{B}$-periods of dp take values in $2\pi\mathbb{Z}$

$$\int_{\mathcal{A}_I} dp = 0, \quad \int_{\mathcal{B}_I} dp = 2\pi n_I, \; n_I \in \mathbb{Z}. \tag{4.6}$$

The integers n_I appearing in the $\mathcal{B}_I$ periods are precisely the mode numbers introduced in (1.5) above. A normalised (i.e. vanishing $\mathcal{A}$-periods) meromorphic differential on Σ is uniquely specified by its singular parts, and so equation (3.8) along with the $\mathcal{A}$-cycle conditions in (4.6) uniquely specify dp as a meromorphic differential on Σ with singular parts

$$\begin{aligned} dp(x^\pm) &= \mp d\left(\frac{\pi\kappa}{x-1}\right) + O\left((x-1)^0\right) \quad \text{as } x \to +1, \\ dp(x^\pm) &= \mp d\left(\frac{\pi\kappa}{x+1}\right) + O\left((x+1)^0\right) \quad \text{as } x \to -1. \end{aligned} \tag{4.7}$$

Because the normalised meromorphic differential dp on Σ is uniquely specified by its singular parts (4.7) it follows that

$$\hat{\sigma}^* dp = -dp.$$

The infinite set of singular points $\{x_k\}$ of the original curve Γ become marked points on the desingularised curve Σ in (4.4) and can be characterised by the following condition

$$p(x_k) = n_k\pi, \qquad n_k \in \mathbb{Z}. \tag{4.8}$$

Singular points can otherwise be seen as degenerated branch cuts, the two branch points of which sit on top of each other. From this point of view, the condition (4.8) corresponds to the degeneration of the $\mathcal{B}$-period condition $\int_{\mathcal{B}_k} dp = 2\pi n_k$ of the meromorphic differential dp for this particular cut. Indeed, given the normalised Abelian differential dp we can

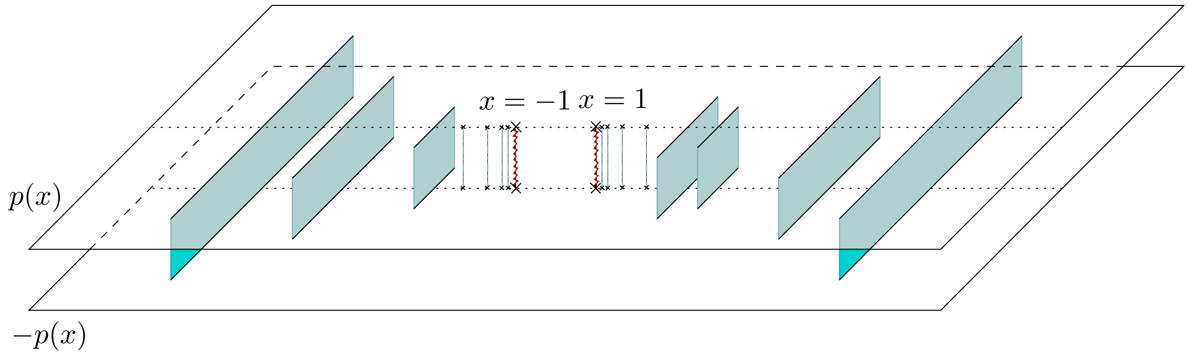

Figure 4: The algebraic curve Σ with its finitely many branch cuts and infinitely many marked points accumulating at $x = \pm 1$.

define its Abelian integral as

$$p(P) = \int_{\infty^+}^{P} dp, \quad P \in \Sigma,$$

If we restrict the integration path to lie on the physical sheet then this function is just the single valued function $p(x)$ itself, i.e. $p(x) = p(x^+)$. The $\mathcal{B}$-period condition on dp now reads (where $Q^+ \in \Sigma$ is a point on the physical sheet arbitrarily close to the cut $\mathcal{C}_k$)

$$\begin{aligned} 2\pi n_k &= \int_{\infty^+}^{Q^+} dp + \int_{Q^+}^{\infty^-} dp = \int_{\infty^+}^{Q^+} dp - \int_{Q^+}^{\infty^-} \hat{\sigma}^* dp \\ &= \int_{\infty^+}^{Q^+} dp - \int_{\hat{\sigma}Q^+}^{\infty^+} dp = \int_{\infty^+}^{Q^+} dp + \int_{\infty^+}^{\hat{\sigma}Q^+} dp \\ \text{i.e. } 2\pi n_k &= p(Q^+) + p(\hat{\sigma}Q^+). \end{aligned}$$

Thus if the cut $\mathcal{C}_k$ degenerates to a singular point then $Q^+ \in \Sigma$ becomes that singular point which satisfies $\hat{\sigma}Q^+ = Q^+$ so that

$$p(Q^+) = n_k \pi.$$

A typical algebraic curve Σ for a finite-gap solution of genus g is represented schematically in Figure 4 with its $g+1$ cuts and marked points (representing singular points of Γ) accumulating at $x = \pm 1$.

As suggested in [4], the spectral curve can thus be understood as the analogue of a Fourier mode decomposition of the string, in the sense that to every integer mode number $n_i \in \mathbb{Z}$ there corresponds either a branch cut (when the mode is excited) with corresponding $\mathcal{B}$-period $\int_{\mathcal{B}_i} dp = 2\pi n_i$ or a singular point (when the mode is switched off) at x_i such that $p(x_i) = n_i \pi$. Turning on a mode, with mode number $m \in \mathbb{Z}$ say, corresponds to opening up the singular point at $p(x) = m\pi$ into a full grown cut, which in turn increases the genus of the curve by one requiring the introduction of a new moduli as we will shortly argue, i.e. the amplitude of the mode m. Now since the algebraic curve is (σ, τ)-invariant, its moduli gives a set of action variables of the theory, which by the previous reasoning forms a complete set. For a general solution we expect the corresponding curve to have infinite genus, but we can view such a solution as a limit of finite-gap solutions.

Finally, we note that the spectral curve $\widehat{\Sigma}$ (4.3) of [3, 4] can be uniquely recovered from the algebraic curve Σ (4.4) of [2] equipped with the meromorphic differential dp simply by keeping the same spectral parameter x but redefining the y coordinate by

$$dp = ydx.$$

This definition makes sense away from branch points of Σ where x can be taken as a local parameter, and can be extended by continuity to branch points where dx has zeroes so that y picks up a pole at branch points. Therefore the specification of the pair (Σ, dp) is equivalent to the specification of the curve $\widehat{\Sigma}$.

4.2 The moduli space of the spectral curve

The curve $\widehat{\Sigma}$, or equivalently the pair (Σ, dp), uniquely determine the spectrum of conserved charges for any finite-gap solution. In the following, we will define a moduli space for this holomorphic data which, when subjected to the appropriate reality conditions, corresponds to the space of allowed values for the conserved charges. In this section it will be convenient to work initially with configurations where only the first set of Virasoro constraints (2.7) are imposed. The two remaining conditions, which are the momentum constraint and the string mass shell condition (2.9), will be imposed at a later stage in our discussion. As discussed above, we will also restrict our attention to variations of the spectral curve for which the left and right Noether charges L and R remain constant.

The required moduli space is a special case of a very general construction due to Krichever and Phong [13] which is reviewed in appendix B. Following [13], the starting point of the construction is a much larger *universal moduli space*, $\mathcal{U}$, of smooth Riemann surfaces of genus g, with N punctures P_α, equipped with two Abelian integrals E and Q which have poles of order n_α and m_α respectively at the punctures. This space has complex dimension $5g - 3 + 3N + \sum_{i=1}^{N}(n_\alpha + m_\alpha)$. Krichever and Phong define a set of local holomorphic coordinates on $\mathcal{U}$ and prove that level sets of certain subsets of these coordinates define smooth foliations of $\mathcal{U}$. As shown in [13], the moduli spaces of the spectral data for a large class of integrable systems can be each described as leaves of such a foliation. Remarkably, the relevant moduli space for the holomorphic data (Σ, dp) also admits precisely such a description.

In the present case the Riemann surface is identified with the algebraic curve Σ of genus $g = K - 1$ defined above and the punctures P_α correspond to the eight points $\{\infty^\pm, 0^\pm, (+1)^\pm, (-1)^\pm\}$ on Σ. The first Abelian integral E is identified with the quasi-momentum $p(P)$ defined in the previous section. As in (3.8), $p(P)$ has simple poles at the points $\{(+1)^\pm, (-1)^\pm\}$ and no other singularities. To make contact with [13] we are led to introduce a second Abelian integral $z(P)$ on Σ, identified with Q in the general construction, which has simple poles at the remaining punctures $\{\infty^\pm, 0^\pm\}$ and no other singularities. Thus we are considering the $N = 8$ case of the construction of [13] with $m_\alpha + n_\alpha = 1$ at each of the eight punctures P_α. In this case, the universal moduli space $\mathcal{U}$ has complex dimension $5g + 29 = 5K + 24$.

As in the previous section Σ and p are subject to several constraints. These include,

- The a- and b-periods (4.6) of the differential dp.
- The asymptotics of dp near the points $\{(+1)^\pm, (-1)^\pm\}$. As explained above, at this stage we are only imposing the first set of Virasoro constraints (2.7). In this case the asymptotics (4.7) are replaced by,

$$\begin{aligned} dp(x^\pm) &= \mp d\left(\frac{\pi\kappa_+}{x-1}\right) + O\left((x-1)^0\right) \quad \text{as } x \to +1, \\ dp(x^\pm) &= \mp d\left(\frac{\pi\kappa_-}{x+1}\right) + O\left((x+1)^0\right) \quad \text{as } x \to -1. \end{aligned} \tag{4.9}$$

- The asymptotics (3.13), of p near the points $\{\infty^\pm, 0^\pm\}$.

In appendix B, we show that the set of constraints described above is equivalent to a collection of level-set conditions for a subset of the holomorphic coordinates on $\mathcal{U}$ introduced in [13]. Further level-set conditions are provided by specifying appropriate residues and periods for the second Abelian integral $z(P)$. In total, we find level-set conditions for exactly $4g + 29$ of the $5g + 29$ holomorphic coordinates on $\mathcal{U}$. This defines a leaf $\mathcal{L}$ in a smooth g-dimensional foliation of the universal moduli space.

In the above discussion we did not assume the hyperelliptic form (4.4) for the curve Σ. In fact, as we show in appendix B, the hyperelliptic property of Σ is actually a consequence of the level-set conditions described above and therefore leads to no further constraints. Finally we can also derive an explicit expression for the second Abelian integral $z(P)$ in terms of the hyperelliptic coordinate $x(P)$. Specifically, we find the formula,

$$z(P) = x(P) + \frac{1}{x(P)}. \tag{4.10}$$

The remaining holomorphic coordinates which are not fixed by the level-set conditions provide holomorphic coordinates on the leaf $\mathcal{L}$. According to [13], these coordinates correspond to certain periods (and residues) of the meromophic differential

$$\alpha = \frac{\sqrt{\lambda}}{4\pi} z dp, \tag{4.11}$$

where we have chosen an appropriate normalisation. The g independent coordinates parameterising the leaf are [13],

$$S_i = \frac{1}{2\pi i}\int_{a_i} \alpha = \frac{1}{2\pi i}\frac{\sqrt{\lambda}}{4\pi}\int_{a_i}\left(x + \frac{1}{x}\right) dp, \quad i = 1, \ldots, g, \tag{4.12}$$

where a_i are the canonical one-cycles on Σ defined in section 4.1 and we have used (4.10). Equivalently one can work with the periods of α around the $K = g + 1$ cycles, $\mathcal{A}_I$, also defined in section 4.1, and choose coordinates,

$$\mathcal{S}_I = \frac{1}{2\pi i}\int_{\mathcal{A}_I}\alpha = \frac{1}{2\pi i}\frac{\sqrt{\lambda}}{4\pi}\int_{\mathcal{A}_I}\left(x+\frac{1}{x}\right)dp, \quad I=1,\ldots,K. \tag{4.13}$$

subject to the constraint,

$$\sum_{I=1}^{K}\mathcal{S}_I = \mathrm{Res}_{\infty^+}\alpha + \mathrm{Res}_{0^+}\alpha = \frac{1}{2}(L-R). \tag{4.14}$$

Thus we have one coordinate $\mathcal{S}_I$ associated with each cut $\mathcal{C}_I$ in the x-plane. The relation between the two sets of cycles implies that $\mathcal{S}_I = S_i$ for $I=i=1,\ldots,g$.

The variables $\mathcal{S}_I$ are known as *filling fractions* [2, 3] and they play an important role in the context of the AdS/CFT correspondence. As in (4.14), their sum is the global charge $J=(L-R)/2$. In the dual spin-chain description, each magnon carries one unit of the charge J and also corresponds to a single Bethe root in the complex rapidity plane. The Bethe roots condense to form cuts in an appropriate thermodynamic limit. The quantity $\mathcal{S}_I$ corresponds to the total $U(1)_J$ charge associated with the cut $\mathcal{C}_I$ and thus to the total number of Bethe roots with the corresponding mode number n_I. The AdS/CFT correspondence therefore suggest that the quantities $\mathcal{S}_I$ (and thus the moduli S_i) should be quantised in integer units. In subsection 5.8, we will find an intriguing hint that this quantisation will emerge naturally on the string theory side of the correspondence.

It is important to emphasise that the quantities $\kappa_\pm$ which appear in the Virasoro constraint (2.7) and the asymptotics (4.9), are not independent parameters in the construction described above. Rather they are non-trivial functions of the coordinates $\{S_i\}$ on the leaf $\mathcal{L}$. These quantities also determine (via (2.8)) the worldsheet momentum, $\mathcal{P}$ and energy, $\mathcal{E}$, which should likewise be thought of as functions on $\mathcal{L}$. Fortunately it is possible to describe the variations of $\mathcal{P}$ and $\mathcal{E}$ with the moduli $\{S_i\}$ quite explicitly. In appendix C we derive the following equations,

$$\delta^{\mathcal{L}}\mathcal{P} = -\sum_{i=1}^{g}k_i\delta^{\mathcal{L}}S_i, \quad \delta^{\mathcal{L}}\mathcal{E} = -\sum_{i=1}^{g}w_i\delta^{\mathcal{L}}S_i, \tag{4.15}$$

using the Riemann bilinear identity. Here k_i and w_i are defined in terms of the periods of the meromorphic differentials dp and dq of the quasi-momentum and quasi-energy respectively,

$$\int_{b_i}dp = 2\pi k_i, \quad \int_{b_i}dq = 2\pi w_i. \tag{4.16}$$

The variation $\delta^{\mathcal{L}}$, corresponds to the exterior derivative on the complex leaf $\mathcal{L}$.

As in (4.6) the closed string boundary condition constrains the k_i to be integers which are related to the mode numbers n_I as $k_i = n_i - n_K$. Thus the k_i are constant on the leaf $\mathcal{L}$ and we can integrate the first equality in (4.15) to obtain,

$$\mathcal{P} = -\sum_{i=1}^{g}k_iS_i + \text{constant}.$$

Using (4.14), we can rewrite $\mathcal{P}$ in terms of the filling fractions $\mathcal{S}_I$ and fix the integration constant by comparison to the pointlike string solution with $\mathcal{S}_I = 0$ for all I. The final result is,

$$\mathcal{P} = -\sum_{I=1}^{K} n_I \mathcal{S}_I. \tag{4.17}$$

Before imposing the momentum constraint and the string mass shell condition, the moduli space of admissible spectral curves Σ is precisely the leaf $\mathcal{L}$ described above. From (4.17) , the momentum constraint implies a further linear condition on the filling fractions,

$$\sum_{I=1}^{K} n_I \mathcal{S}_I = 0. \tag{4.18}$$

As before the string mass shell condition identifies the string energy Δ as $\sqrt{2\sqrt{\lambda}\mathcal{E}}$ The second equality in (4.15) then provides a set of first order differential equations for Δ as a function of the moduli $\{S_i\}$,

$$\Delta \frac{\partial \Delta}{\partial S_i} = -\frac{\sqrt{\lambda}}{2\pi} \int_{b_i} dq.$$

In contrast to the integers k_i, the b-periods w_i of dq are not constant along the leaf $\mathcal{L}$ and integration of these equations is non-trivial.

4.3 Reality conditions

The reality condition (3.16) on the monodromy matrix implies that the curve Γ is invariant under the following anti-holomorphic involution

$$\hat{\tau} : \Gamma \to \Gamma, (x, y) \mapsto (\bar{x}, \bar{y}^{-1}). \tag{4.19}$$

In terms of the representation of Γ as a 2-sheeted ramified cover of the x-plane the effect of this anti-holomorphic involution is simply to map both sheets to themselves by $x \mapsto \bar{x}$. The two involutions (4.19) and (4.2) together generate a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group of involutions on Γ such that $\hat{\sigma}\hat{\tau} = \hat{\tau}\hat{\sigma}$. Since in going from the curve Γ in (4.1) to the fully desingularised curve Σ in (4.4) we haven't modified the spectral parameter x, the desingularised curve Σ is still invariant under the anti-holomorphic involution which takes both sheets to themselves by $x \mapsto \bar{x}$, so that the branch points u_I, v_I defined in (4.4) should either be both real ($u_I = \bar{u}_I$, $v_I = \bar{v}_I$) or form a complex conjugate pair ($u_I = \bar{v}_I$).

It follows from its definition that the homology basis $\{a_i, b_i\}_{i=1}^{g}$ has the following property under the action of the anti-holomorphic involution $\hat{\tau}$

$$\hat{\tau} a_i = -a_i, \quad \hat{\tau} b_i = b_i, \tag{4.20}$$

where the equalities in these expressions are to be understood modulo cycles homotopic to zero. Note also that the basis of holomorphic differentials of Σ is such that $\overline{\hat{\tau}^* \omega_i} = -\omega_i$ since $\overline{\hat{\tau}^* \omega_i}$ are holomorphic 1-forms (because the local parameter x is such that $\hat{\tau}^* x = \bar{x}$) and

$$\delta_{ij} = \int_{a_i} \omega_j = \int_{\hat{\tau} a_i} \hat{\tau}^* \omega_j = -\int_{a_i} \hat{\tau}^* \omega_j.$$

Thus it follows that the periodic matrix $\Pi_{ij} = \int_{b_i} \omega_j$ of Σ is pure imaginary $\overline{\Pi} = -\Pi$ because

$$\overline{\Pi_{ij}} = \int_{b_i} \overline{\omega_j} = \int_{\hat{\tau} b_i} \overline{\hat{\tau}^* \omega_j} = -\int_{b_i} \omega_j = -\Pi_{ij}.$$

The reality condition on the differential introduced in (4.11) is $\overline{\hat{\tau}^* \alpha} = \alpha$, from which the reality condition on the filling fractions (4.13) follows

$$\overline{\mathcal{S}_I} = -\frac{1}{2\pi i} \int_{\mathcal{A}_I} \bar{\alpha} = -\frac{1}{2\pi i} \int_{\mathcal{A}_I} \hat{\tau}^* \alpha = -\frac{1}{2\pi i} \int_{\hat{\tau} \mathcal{A}_I} \alpha = \mathcal{S}_I, \quad I = 1, \dots, K.$$

5. Finite-gap solutions

In the previous sections we have seen how from a given solution to the equations of motion one can construct a flat connection $J(x, \sigma, \tau)$, and from this obtain a non-singular algebraic curve carrying the action variables of the solution in question. In section 5.1 we identify the remaining algebro-geometric data which uniquely characterises the dynamics of the solution, i.e. the angle variables. We then explain in section 5.2 the method of finite-gap integration for recovering the explicit solution corresponding to a given set of algebro-geometric data [19−21, 23]. In the following section we will use various results proved for example in [19] so to avoid cluttering the main argument we will simply refer to specific pages in this reference for the proofs.

5.1 Identifying the algebro-geometric data

We start by describing the dynamics of the monodromy matrix which from the relation (3.7) is encoded in its eigenvectors. By definition of the spectral curve there is a unique eigenvector of $\Omega(x, \sigma, \tau)$ corresponding to any regular point $P \in \Gamma$, and it can be shown (see the Proposition p131-132 in [19] and especially the remarks following it) that this is also the case when P is a branch point. Thus the desingularised curve Σ has a unique eigenvector $\boldsymbol{h}(P, \sigma, \tau)$ at each point $P \in \Sigma$, which we normalise by setting its first component to one for convenience, i.e. $h_1(P, \sigma, \tau) = 1$. The second component h_2 on Σ can be shown to have exactly $g + 1$ poles in the present case (see the Proposition p134 in [19]). At the points $\infty^+, \infty^- \in \Sigma$ above $x_0 = \infty$ the eigenvector is proportional to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively, so the second component $h_2(P, \sigma, \tau)$ of the normalised eigenvector has a (simple) zero at ∞^+ and one of its $g + 1$ poles is at ∞^-. The remaining g poles define the *dynamical divisor* which we denote by $\gamma(\sigma, \tau) = \prod_{i=1}^g \gamma_i(\sigma, \tau)$. It is easy to see that the algebro-geometric data $\{\Sigma, dp, \gamma(\sigma, \tau), \infty^\pm\}$ is invariant under the residual gauge group (section 3) and so we have a well defined map

$$[\Omega(x, \sigma, \tau)] \mapsto \left\{\Sigma, dp, \gamma(\sigma, \tau), \infty^\pm\right\}, \tag{5.1}$$

where $[\cdot]$ denotes the equivalence class under residual gauge transformations. But now by the Riemann-Roch theorem there is up to multiplication by a function independent of P a unique meromorphic function on Σ with pole divisor in $\gamma(\sigma, \tau)\infty^-$ and with a zero at ∞^+. Thus the divisor $\left(\gamma(\sigma, \tau)\infty^-\right)^{-1} \infty^+$ on Σ uniquely specifies the normalised eigenvector

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$\boldsymbol{h}(P,\sigma,\tau)$ up to left multiplication by a diagonal matrix $\text{diag}(1,d(\sigma,\tau))$ and hence also completely specifies the monodromy matrix up to a similarity transformation $\Omega(x,\sigma,\tau)\to \text{diag}(1,d(\sigma,\tau))\Omega(x,\sigma,\tau)\text{diag}(1,d(\sigma,\tau))^{-1}$. Therefore the map (5.1) is injective.

To determine the dynamics of the connection $J(x,\sigma,\tau)$ we first notice that the zero-curvature equation (3.2) is equivalent to the consistency condition for the following linear system

$$(d - J(x,\sigma,\tau))\,\boldsymbol{\psi} = 0, \tag{5.2}$$

The Lax connection $J(x,\sigma,\tau)$ can be recovered at a generic value of x from the solution $\boldsymbol{\psi}(P,\sigma,\tau)$ to the auxiliary linear problem by

$$J(x,\sigma,\tau) = (d\Psi(x,\sigma,\tau))\,\Psi(x,\sigma,\tau)^{-1}, \tag{5.3}$$

where $\Psi(x,\sigma,\tau) = (\boldsymbol{\psi}(x^+),\boldsymbol{\psi}(x^-))$, and thus we can focus on describing the dynamics of the solution $\boldsymbol{\psi}(P,\sigma,\tau)$ to the auxiliary linear problem. Now by virtue of the relation (3.6), the operators $(d - J(x,\sigma,\tau))$ and $\Omega(x,\sigma,\tau)$ can be simultaneously diagonalised, and hence the vector $\boldsymbol{\psi}(P,\sigma,\tau)$ in (5.2) can be taken as a multiple of the normalised eigenvector

$$\boldsymbol{\psi}(P,\sigma,\tau) = \varphi(P,\sigma,\tau)\boldsymbol{h}(P,\sigma,\tau). \tag{5.4}$$

Taking the first component of the auxiliary linear problem (5.2) yields a system of equations for the scalar function $\varphi(P,\sigma,\tau)$, namely

$$d\varphi(P,\sigma,\tau) = \big(J(x,\sigma,\tau)\boldsymbol{h}(P,\sigma,\tau)\big)_1\varphi(P,\sigma,\tau), \tag{5.5}$$

Now since the Lax connection $J(x,\sigma,\tau)$ has poles at the constant values $x=\pm1$ and the normalised eigenvector is holomorphic in a neighbourhood of every point above $x=\pm1$, equation (5.5) implies that the scalar function $\varphi(P,\sigma,\tau)$ has essential singularities at every point on Σ above $x=\pm1$ which are shown in Appendix D to be of the form

$$\begin{aligned}\varphi(x^{\pm},\sigma,\tau) &= O(1)\ \exp\left\{\mp\frac{i\kappa_+}{2}\frac{\sigma+\tau}{x-1}\right\}, \quad \text{as } x\to 1,\\ \varphi(x^{\pm},\sigma,\tau) &= O(1)\ \exp\left\{\mp\frac{i\kappa_-}{2}\frac{\sigma-\tau}{x+1}\right\}, \quad \text{as } x\to -1,\end{aligned} \tag{5.6}$$

where $O(1)$ denotes a function holomorphic in a neighbourhood of $x=\pm1$. Because of (3.10) and the gauge redundancy (3.18), we have in a general residual gauge $J(\infty,\sigma,\tau) = d\tilde{g}\tilde{g}^{-1}$ where $\tilde{g}(\sigma,\tau)$ is diagonal, so that the constant pole of h_2 at ∞^- does not give rise to any essential singularities in $\varphi(P,\sigma,\tau)$. The function $\varphi(P,\sigma,\tau)$ also has zeroes at the dynamical divisor $\gamma(\sigma,\tau)$ (see [19] p151-152) and if we consider the solution to (5.2) with initial condition[9] $\varphi(P,0,0)=1$ then $\varphi(P,\sigma,\tau)$ has poles at $\gamma(0,0)$. But a meromorphic function on $\Sigma\setminus\{x=\pm1\}$ with pole divisor in $\gamma(0,0)$ and essential singularities at the

[9]Normalising φ such that $\varphi(P,0,0)=1$ is always possible because one can divide the solution φ to the auxiliary linear problem (5.2) by an arbitrary function $f(P)$ of $P\in\Sigma$ only. Note that this scaling by $f(P)$ has no effect on the current $J(x,\sigma,\tau)$ reconstructed using (5.3). Choosing $f(P)=\varphi(P,0,0)$ yields the required initial condition on φ.

punctures $x = \pm 1$ of the prescribed form (5.6) is unique[10] up to multiplication by a function independent of $P \in \Sigma$. Therefore the first component $\psi_1 = \varphi$ of $\boldsymbol{\psi}(P, \sigma, \tau)$ is uniquely specified up to a multiplicative $c(\sigma, \tau)$ by the divisor $\gamma(0,0)$ and its behaviour (5.6) at $x = \pm 1$. Likewise, the second component $\psi_2 = \varphi h_2$ is uniquely specified up to a multiplicative $d(\sigma, \tau)$ by: the behaviour (5.6) of φ at $x = \pm 1$, the fact that ψ_2 has a zero at ∞^+ and that its pole divisor is contained in $\gamma(0,0)\infty^-$. Therefore the set of algebro-geometric data for the solution $\boldsymbol{\psi}(P, \sigma, \tau)$ to the auxiliary linear problem (5.2) can be taken to be $\{\Sigma, dp, \gamma(0,0), \infty^\pm, S_\pm(P, \sigma, \tau)\}$ where $S_\pm(P, \sigma, \tau)$ are the singular parts defined in (D.3) and it is easily seen to be invariant under residual gauge transformations ($\boldsymbol{\psi} \to \tilde{g}\boldsymbol{\psi}$). Hence we have a well defined injective map

$$[J(x, \sigma, \tau)] \mapsto \{\Sigma, dp, \gamma(0,0), \infty^\pm, S_\pm(P, \sigma, \tau)\}. \tag{5.7}$$

Note that the singular parts are specific to the choice of analytic behaviour of $J(x, \sigma, \tau)$ in x and that the pair of punctures $\infty^\pm \in \Sigma$ is specific to the gauge fixing choice (3.12) and neither of them depends on the solution considered. So since this extra data is common to all solutions we may omit it when specifying the algebro-geometric data so that the injective map (5.7) boils down to

$$[J(x, \sigma, \tau)] \mapsto \{\Sigma, dp, \gamma(0,0)\}. \tag{5.8}$$

The divisor $\gamma(0,0)$ being of degree g it lives in the g-th symmetric product $S^g(\Sigma) = \Sigma^g / S_g$ of the curve Σ which is in bijection with the Jacobian $J(\Sigma)$ of Σ by means of the Abel map

$$\begin{aligned} \boldsymbol{\mathcal{A}} : S^g(\Sigma) &\to J(\Sigma) \\ \prod_{i=1}^{g} P_i &\mapsto 2\pi \sum_{i=1}^{g} \int_{\infty^+}^{P_i} \boldsymbol{\omega}. \end{aligned} \tag{5.9}$$

Therefore the algebro-geometric data $\{\Sigma, dp, \gamma(0,0)\}$ corresponding to a generic solution specifies a point on the Jacobian bundle $\mathcal{M}_{\mathbb{C}}^{(2g)}$ over the leaf $\mathcal{L}$ which we defined in section 4

$$J(\Sigma) \to \mathcal{M}_{\mathbb{C}}^{(2g)} \to \mathcal{L}, \tag{5.10}$$

where the fibre over any point Σ in $\mathcal{L}$ is the Jacobian $J(\Sigma)$ of the corresponding curve Σ. So in effect the map (5.8) constructed in this section takes the orbit under residual gauge transformations of any solution of the equations of motion (2.4) (note that not every point on this orbit is a solution) to a single point on $\mathcal{M}_{\mathbb{C}}^{(2g)}$.

5.2 Reconstruction formulae

In the previous section we have identified the set of algebro-geometric data which uniquely specifies (up to residual gauge transformations) a solution $J(x, \sigma, \tau)$ to the zero-curvature equations, as exhibited by the injective map (5.8). In this section we want to construct the

[10] Indeed, suppose there are two such functions, then their quotient is meromorphic on the genus g Riemann surface Σ and has at most $g = \deg \gamma(0,0)$ poles which must be independent of $P \in \Sigma$ by the Riemann-Roch theorem.

(left) inverse of this map (5.8), namely

$$\{\Sigma, dp, \gamma(0,0)\} \mapsto [J(x,\sigma,\tau)]\,. \tag{5.11}$$

In other words, starting from a given set of admissible algebro-geometric data (i.e. a point on $\mathcal{M}_{\mathbb{C}}^{(2g)}$) — a curve Σ of the form (4.4) equipped with an Abelian differential dp and a constant divisor $\gamma(0,0)$ of degree g, along with the pair of punctures $\infty^{\pm} \in \Sigma$ above $x_0 = \infty \in \mathbb{CP}^1$ and the singular parts $S_{\pm}(P,\sigma,\tau)$ — we want to recover the Lax connection $J(x,\sigma,\tau)$ corresponding to this set of data modulo residual gauge transformations.

The components ψ_1, ψ_2 of $\boldsymbol{\psi}$ are meromorphic functions on $\Sigma \setminus \{x = \pm 1\}$ which were shown in the previous section to be uniquely specified (up to multiplicative functions independent of $P \in \Sigma$) by their respective divisors $(\psi_1) \geq \gamma(0,0)^{-1}$ and $(\psi_2) \geq (\gamma(0,0)\infty^-)^{-1}\infty^+$ and by their common behaviour (5.6) near the essential singularities at $x = \pm 1$. We can scale the eigenvector $\boldsymbol{\psi}$ by a function $k_-(P)$ independent of σ and τ so we let $\hat{\gamma}$ be a positive divisor of degree $\deg \hat{\gamma} = g+1$ equivalent[11] to $\gamma(0,0)\infty^-$ and define $k_-(P)$ as the unique function with divisor $(\gamma(0,0)\infty^-)\,\hat{\gamma}^{-1}$ normalised by $k_-(\infty^+) = 1$. After scaling by $k_-(P)$ the divisors of the components of $\boldsymbol{\psi}$ satisfy

$$(\psi_1) \geq \hat{\gamma}^{-1}\infty^-, \quad (\psi_2) \geq \hat{\gamma}^{-1}\infty^+, \tag{5.12}$$

whereas their behaviour near the essential singularities at $x = \pm 1$ is unchanged. Functions with such defining properties are known as Baker-Akhiezer functions and can be constructed on the Riemann surface Σ of genus g with the help of Riemann θ-functions [19, 23], so by uniqueness these constructed functions will have to be equal to ψ_1, ψ_2 up to a multiplicative function independent of $P \in \Sigma$; we fix these multiplicative functions of σ, τ in the definitions of ψ_1, ψ_2 by choosing the following normalisation $\psi_1(\infty^+) = \psi_2(\infty^-) = 1$. Using the standard expressions for Baker-Akhiezer functions in terms of Riemann θ-functions (see for instance section 2.7 in [23]), the components of $\boldsymbol{\psi}(P,\sigma,\tau)$ are given by

$$\psi_1(P,\sigma,\tau) = k_-(P)\frac{\theta\left(\boldsymbol{\mathcal{A}}(P) + \boldsymbol{k}\sigma + \boldsymbol{w}\tau - \boldsymbol{\zeta}_{\gamma(0,0)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^+) - \boldsymbol{\zeta}_{\gamma(0,0)}\right)}{\theta\left(\boldsymbol{\mathcal{A}}(P) - \boldsymbol{\zeta}_{\gamma(0,0)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^+) + \boldsymbol{k}\sigma + \boldsymbol{w}\tau - \boldsymbol{\zeta}_{\gamma(0,0)}\right)} \times \exp\left(\frac{i\sigma}{2\pi}\int_{\infty^+}^P dp + \frac{i\tau}{2\pi}\int_{\infty^+}^P dq\right), \tag{5.13a}$$

$$\psi_2(P,\sigma,\tau) = k_+(P)\frac{\theta\left(\boldsymbol{\mathcal{A}}(P) + \boldsymbol{k}\sigma + \boldsymbol{w}\tau - \boldsymbol{\zeta}_{\gamma'(0,0)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^-) - \boldsymbol{\zeta}_{\gamma'(0,0)}\right)}{\theta\left(\boldsymbol{\mathcal{A}}(P) - \boldsymbol{\zeta}_{\gamma'(0,0)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^-) + \boldsymbol{k}\sigma + \boldsymbol{w}\tau - \boldsymbol{\zeta}_{\gamma'(0,0)}\right)} \times \exp\left(\frac{i\sigma}{2\pi}\int_{\infty^-}^P dp + \frac{i\tau}{2\pi}\int_{\infty^-}^P dq\right), \tag{5.13b}$$

where the ingredients for these formulae are defined as follows:

[11] Two divisors D_1, D_2 are said to be equivalent and we write $D_1 \sim D_2$ if there exists a meromorphic function f with divisor $(f) = D_1 D_2^{-1}$.

- The function $\theta : \mathbb{C}^g \to \mathbb{C}$ is the Riemann θ-function associated with Σ defined for $\boldsymbol{z} \in \mathbb{C}^g$ by

$$\theta(\boldsymbol{z}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^g} \exp\left\{i\langle \boldsymbol{m}, \boldsymbol{z}\rangle + \pi\, i\langle \Pi\boldsymbol{m}, \boldsymbol{m}\rangle\right\},$$

  where $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \sum_{i=1}^g x_i y_i$ and Π is the period matrix (4.5). It has the following important property under translation by vectors $2\pi\boldsymbol{n} + 2\pi\Pi\boldsymbol{m} \in 2\pi\mathbb{Z}^g + 2\pi\Pi\mathbb{Z}^g$

$$\theta(\boldsymbol{z} + 2\pi\boldsymbol{n} + 2\pi\Pi\boldsymbol{m}) = \exp\left\{-i\langle \boldsymbol{m}, \boldsymbol{z}\rangle - \pi i\langle \Pi\boldsymbol{m}, \boldsymbol{m}\rangle\right\}\theta(\boldsymbol{z}). \tag{5.14}$$

- The divisor $\gamma'(0,0)$ is the unique positive divisor such that

$$\gamma'(0,0)\infty^+ \sim \gamma(0,0)\infty^- \sim \hat{\gamma}. \tag{5.15}$$

  Then $k_+(P)$ is the unique function with divisor $(k_+) = (\gamma'(0,0)\infty^+)\,\hat{\gamma}^{-1}$ normalised by the condition $k_+(\infty^-) = 1$, just as $k_-(P)$ is the unique function with divisor $(k_-) = (\gamma(0,0)\infty^-)\,\hat{\gamma}^{-1}$ normalised by $k_-(\infty^+) = 1$.

- The function $\boldsymbol{\mathcal{A}} : \Sigma \to J(\Sigma)$ is the Abel map with base point P_0

$$\begin{aligned} \boldsymbol{\mathcal{A}} : \Sigma &\to J(\Sigma) \\ P &\mapsto 2\pi\int_{P_0}^{P} \boldsymbol{\omega}. \end{aligned}$$

  This map is an injective holomorphic map by Abel's theorem. As in section 5.1 it can be extented (5.9) to act on divisors of degree g by requiring that it be a morphism. This extended map is still injective by Abel's theorem but is also surjective by Jacobi's inversion theorem, and so the map (5.9) defines a bijection between positive divisors of degree g and points on $J(\Sigma)$ as already pointed out in section 5.1.

  We define $\zeta_D = \boldsymbol{\mathcal{A}}(D) + \boldsymbol{\mathcal{K}}$ for any divisor of degree $\deg D = g$ where $\boldsymbol{\mathcal{K}}$ is the vector of Riemann's constants. Since $\zeta_{\gamma'(0,0)} = \zeta_{\gamma(0,0)} - 2\pi\int_{\infty^-}^{\infty^+}\boldsymbol{\omega}$, the Abel map only enters in (5.13) through the expression $\boldsymbol{\mathcal{A}}(P) - \zeta_{\gamma(0,0)}$ and it follows that the formulae (5.13) are base-point independent: changing the base point of the Abel map $\boldsymbol{\mathcal{A}}$ can be absorbed in a change of the initial conditions $\gamma(0,0)$ of the solution and so without loss of generality we choose $P_0 = \infty^+$. We also note that the vector $\zeta_{\gamma(0,0)}$ is almost arbitrary, the only requirement being that the function $\theta\left(\boldsymbol{\mathcal{A}}(P) - \zeta_{\gamma(0,0)}\right)$ should not vanish identically, which is equivalent to the requirement that the divisor $\gamma(0,0)$ be non-special [23].

- Apart from the holomorphic differentials $\boldsymbol{\omega}$ the expressions in (5.13) involve various other Abelian differentials. Firstly, dp is the normalised second kind Abelian differential of the quasi-momentum and is uniquely specified by its pole structure (4.9) at $x = \pm 1$. Secondly, dq is the normalised second kind Abelian differential of the quasi-energy $q(x)$ uniquely specified by its poles at $x = \pm 1$. As for the differential

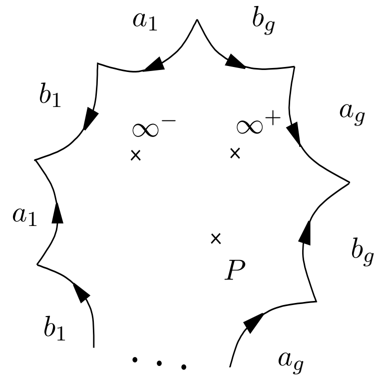

Figure 5: Normal form Σ_{cut} of the Riemann surface Σ of genus g.

dp in (4.9), imposing only the first set of Virasoro constraints (2.7), the asymptotics of dq read,

$$\begin{aligned} dq(x^{\pm}) &= \mp d\left(\frac{\pi\kappa_+}{x-1}\right) + O\left((x-1)^0\right) \quad \text{as } x \to +1, \\ dq(x^{\pm}) &= \pm d\left(\frac{\pi\kappa_-}{x+1}\right) + O\left((x+1)^0\right) \quad \text{as } x \to -1. \end{aligned} \tag{5.16}$$

The relation among the pole part of the quasi-energy (5.16), the pole part of the quasi-momentum in (4.9) and the singular parts $S_{\pm}$ of the problem in (D.3) is

$$\sigma dp + \tau dq \underset{x\to\pm 1}{\sim} 2\pi i dS_{\pm}. \tag{5.17}$$

The vectors in $\mathbb{C}^g$ of b-periods of these Abelian differentials

$$\boldsymbol{k} = \frac{1}{2\pi}\int_{\boldsymbol{b}} dp, \quad \boldsymbol{w} = \frac{1}{2\pi}\int_{\boldsymbol{b}} dq$$

being generically non-zero, the Abelian integrals $\int^P dp$ and $\int^P dq$ define multi-valued functions on Σ. Nevertheless it is straightforward to check using the property (5.14) of θ-functions that the formulae in (5.13) do define single-valued functions ψ_1 and ψ_2 on Σ: if $P \in \Sigma$ goes around an arbitrary cycle $\gamma = \sum_{i=1}^{g}[N_i a_i + M_i b_i] \in H_1(\Sigma, \mathbb{Z})$, $\boldsymbol{N}, \boldsymbol{M} \in \mathbb{Z}^g$ then the expressions in (5.13) are unchanged.

- Since Abelian integrals are multi-valued it is preferable to work with specific branches of these functions by cutting up the Riemann surface Σ along the homology basis $a_i, b_i, i = 1, \dots, g$ to form a simply connected domain Σ_{cut} called the normal form of Σ, see Figure 5. The Abelian integrals $\boldsymbol{\mathcal{A}}(P)$, $\int^P dp$, and $\int^P dq$ are now well defined on Σ_{cut} where all the paths of integration should lie in the interior of Σ_{cut}.

- At this stage the vector $\boldsymbol{\psi}$ is determined only up to a residual gauge transformation $\boldsymbol{\psi} \to \tilde{g}\boldsymbol{\psi}$ where $\tilde{g} = \text{diag}\,(c(\sigma,\tau), d(\sigma,\tau))$, but we will show in the remainder of this section that no further residual gauge transformation is actually required for the reconstruction of the current j.

Since $\gamma(\sigma,\tau)\infty^-$ is the divisor of zeroes of the first component ψ_1 it follows from the formula (5.13a) for ψ_1 and using Riemann's theorem on the zeroes of the θ-function that

$$\boldsymbol{\mathcal{A}}(\gamma(\sigma,\tau)) - \boldsymbol{\mathcal{A}}(\gamma(0,0)) = -\boldsymbol{k}\sigma - \boldsymbol{w}\tau. \tag{5.18}$$

In other words, the motion of the dynamical divisor $\gamma(\sigma,\tau)$ is linearised on the Jacobian $J(\Sigma)$ by the Abel map. Notice that since the right hand side of (5.18) is uniquely determined by the singular parts $S_\pm(P)$ of the problem, it follows that the set of algebro-geometric data (5.1) for $\Omega(x,\sigma,\tau)$ is actually equivalent to the set of algebro-geometric data (5.7) for $J(x,\sigma,\tau)$.

The connection $J(x,\sigma,\tau)$ can now be reconstructed from the formula (5.3) using the expressions (5.13) for the vector $\boldsymbol{\psi}$ up to a residual gauge (3.18) with diagonal parameter $\tilde{g}(\sigma,\tau)$. In order to obtain expressions for the components j_0, j_1 of the current j we must first show that the reconstructed connection $J(x,\sigma,\tau)$ in (5.3) takes the original form (3.1) for some current j. In section 5.1 the behaviour of the eigenvector $\boldsymbol{\psi}$ near the essential singularities at $x=\pm 1$ was found to be

$$\boldsymbol{\psi}(x^\pm) \underset{x\to 1}{\sim} O(1) e^{\mp\frac{i\kappa_+\sigma^+}{x-1}}, \quad \boldsymbol{\psi}(x^\pm) \underset{x\to -1}{\sim} O(1) e^{\pm\frac{i\kappa_-\sigma^-}{x+1}},$$

from which we may write the behaviour of the matrix $\Psi(x) = (\boldsymbol{\psi}(x^+), \boldsymbol{\psi}(x^-))$ near $x=\pm 1$ as follows

$$\begin{aligned}
\Psi(x,\sigma,\tau) &= \left(\Psi_0(\sigma,\tau) + \sum_{s=1}^\infty \Psi_s(\sigma,\tau)(x-1)^s\right) e^{\frac{i\kappa_+\sigma^+}{1-x}\sigma_3} \quad \text{as } x\to 1,\\
\Psi(x,\sigma,\tau) &= \left(\Phi_0(\sigma,\tau) + \sum_{s=1}^\infty \Phi_s(\sigma,\tau)(x+1)^s\right) e^{\frac{i\kappa_-\sigma^-}{1+x}\sigma_3} \quad \text{as } x\to -1,
\end{aligned} \tag{5.19}$$

where $\sigma_3 = \mathrm{diag}(1,-1)$ is the third Pauli matrix and $\sigma^\pm$ are the light-cone coordinates on the string worldsheet introduced in (2.1). It is straightforward to derive from these expansions the following asymptotics near $x=1$

$$\begin{cases} (\partial_+\Psi)\,\Psi^{-1} = \dfrac{i\kappa_+}{1-x}\left(\Psi_0\sigma_3\Psi_0^{-1}\right) + O(1) \\ (\partial_-\Psi)\,\Psi^{-1} = O(1) \end{cases} \quad \text{as } x\to 1$$

and likewise near $x=-1$,

$$\begin{cases} (\partial_+\Psi)\,\Psi^{-1} = O(1) \\ (\partial_-\Psi)\,\Psi^{-1} = \dfrac{i\kappa_-}{1+x}\left(\Phi_0\sigma_3\Phi_0^{-1}\right) + O(1) \end{cases} \quad \text{as } x\to -1.$$

However we also find from (5.13) that $\Psi = \mathbf{1} + O\left(\frac{1}{x}\right)$ as $x\to\infty$ due to the special normalisation of the vector $\boldsymbol{\psi}$ defined in (5.13), so that

$$(\partial_\pm\Psi)\,\Psi^{-1} = O\left(\frac{1}{x}\right) \quad \text{as } x\to\infty.$$

Thus the above asymptotics at $x = \pm 1, \infty$ take the following form

$$(\partial_+ \Psi)\, \Psi^{-1} = J_+(x) + O(1), \quad (\partial_- \Psi)\, \Psi^{-1} = J_-(1) + O(1) \quad \text{as } x \to 1 \tag{5.20a}$$

$$(\partial_+ \Psi)\, \Psi^{-1} = J_+(-1) + O(1), \quad (\partial_- \Psi)\, \Psi^{-1} = J_-(x) + O(1) \quad \text{as } x \to -1 \tag{5.20b}$$

$$(\partial_\pm \Psi)\, \Psi^{-1} = J_\pm(\infty) + O\left(\frac{1}{x}\right) \quad \text{as } x \to \infty, \tag{5.20c}$$

where the matrices $J_\pm(x)$ here have been defined as

$$\begin{aligned} J_+(x) &= \frac{i\kappa_+}{1-x} \left(\Psi_0 \sigma_3 \Psi_0^{-1}\right), \\ J_-(x) &= \frac{i\kappa_-}{1+x} \left(\Phi_0 \sigma_3 \Phi_0^{-1}\right). \end{aligned} \tag{5.21}$$

To show that these are in fact the light-cone components $J_0 \pm J_1$ of the Lax connection $J(x)$ we use a standard argument based on the uniqueness of the Baker-Akhiezer function (see for instance [23]) and consider the following vector-valued functions on Σ

$$\boldsymbol{f}_\pm(P) = (\partial_\pm - J_\pm(x))\, \boldsymbol{\psi}(P) = \left[(\partial_\pm \Psi(x))\, \Psi(x)^{-1} - J_\pm(x)\right] \boldsymbol{\psi}(P),$$

where $\hat{\pi}(P) = x$. By their definition, on $\Sigma \backslash \{x = \pm 1\}$ the components of the vectors $\boldsymbol{f}_\pm(P)$ have exactly the same *constant* poles as $\boldsymbol{\psi}(P)$ at $\hat{\gamma}$ as well as the same *constant* zeroes as the components of $\boldsymbol{\psi}(P)$ at $\infty^\pm$ (see (5.12)) using the fact that $J_\pm(\infty) = 0$. Also from their definition and using the asymptotics at $x = \pm 1$ in (5.20a) and (5.20b), these vectors have essential singularities at $x = \pm 1$ of exactly the same form as that of the vector $\boldsymbol{\psi}$. These were the properties which determined the Baker-Akhiezer vector $\boldsymbol{\psi}(P)$ uniquely on Σ up to multiplication by a diagonal matrix independent of $P \in \Sigma$, and thus by the uniqueness of the Baker-Akhiezer vector the following must be true

$$\boldsymbol{f}_\pm(P) = D(\sigma, \tau) \boldsymbol{\psi}(P),$$

for some diagonal matrix $D(\sigma, \tau)$ independent of $P \in \Sigma$. But the asymptotics at $x = \infty$ in (5.20c) now show that in fact $D(\sigma, \tau)$ must be zero, so we conclude

$$\boldsymbol{f}_\pm(P) \equiv 0.$$

Going back to the definition of these vectors this implies

$$J_\pm(x) = (\partial_\pm \Psi(x))\, \Psi(x)^{-1}. \tag{5.22}$$

In other words the connection defined in (5.21) is exactly the reconstructed Lax connection, and so the latter is indeed of the form (3.1) once we have completely fixed the residual gauge by imposing the final condition $J(\infty) = 0$ (obtained from (3.10)). We can now read off from (5.21) the reconstruction formula for the light-cone components of the current j

$$\begin{aligned} j_+(\sigma, \tau) &= i\kappa_+ \left(\Psi_0 \sigma_3 \Psi_0^{-1}\right), \\ j_-(\sigma, \tau) &= i\kappa_- \left(\Phi_0 \sigma_3 \Phi_0^{-1}\right), \end{aligned} \tag{5.23}$$

where Ψ_0 and Φ_0 are the matrices defined in terms of the reconstructed matrix Ψ through equation (5.19). The components of the reconstructed current (5.23) can alternatively be written more compactly as follows

$$j_\pm(\sigma,\tau) = i\kappa_\pm \lim_{x\to\pm 1} \left(\Psi(x,\sigma,\tau)\sigma_3\Psi(x,\sigma,\tau)^{-1}\right). \tag{5.24}$$

One can easily check that these reconstructed currents satisfy the first set of Virasoro constraints (2.7) since $j_\pm^2 = -\kappa_\pm^2 \mathbf{1}$ so that

$$\text{tr } j_\pm^2 = -\kappa_\pm^2 \text{ tr } \mathbf{1} = -2\kappa_\pm^2.$$

Also, before having imposed any reality conditions on the algebro-geometric data the reconstructed current (5.24) takes values in $\mathfrak{sl}(2,\mathbb{C})$ since it is obviously invertible and traceless,

$$\text{tr } j_\pm = i\kappa_\pm \text{ tr } \sigma_3 = 0.$$

We can write out the components of the reconstructed current (5.24) even more explicit in terms of θ-functions starting from the expression (5.13) for the matrix of Baker-Akhiezer vectors $\Psi(x,\sigma,\tau)$ in terms of θ-functions. The final expression for the light-cone components of the reconstructed $\mathrm{SU}(2)_R$ current j then reads

$$j_\pm(\sigma,\tau) = e^{\left(\frac{i}{2}\bar{\theta}_0 - \frac{i}{2}\int_{\infty^-}^{\infty^+} d\mathcal{Q}\right)\sigma_3} \Theta(\pm 1,\sigma,\tau)\,(i\kappa_\pm\sigma_3)\,\Theta(\pm 1,\sigma,\tau)^{-1} e^{-\left(\frac{i}{2}\bar{\theta}_0 - \frac{i}{2}\int_{\infty^-}^{\infty^+} d\mathcal{Q}\right)\sigma_3}, \tag{5.25}$$

where the notation in this expression is as follows:

- After reconstructing the current (5.24), the only part of the residual gauge transformation that is still unfixed is conjugation by constant diagonal matrices corresponding precisely to the $U(1)_R$ subgroup of the physical $\mathrm{SU}(2)_R$ symmetry (2.12) of the action that preserves the level set $Q_R = \frac{1}{2i}R\sigma_3$ (although at this stage, before having imposed reality conditions, we are really dealing with a $\mathbb{C}^*$ subgroup of an $SL(2,\mathbb{C})_R$ transformation acting on complexified currents j). All the other spurious gauge redundancy introduced by the zero-curvature formulation (3.2) of the equations of motion has been fixed. This undetermined $\mathbb{C}^*$ conjugation matrix can be expressed in terms of a single arbitrary constant $\bar{\theta}_0 \in \mathbb{C}$ as $e^{\frac{i}{2}\bar{\theta}_0\sigma_3}$.

- We can combine the Abelian differentials dp and dq into the following second kind Abelian differential
$$d\mathcal{Q}(\sigma,\tau) = \frac{1}{2\pi}(\sigma dp + \tau dq),$$
which plays a special role in the dynamics of finite-gap solutions. Referring to equation (5.17), it can otherwise be defined as the unique normalised second kind Abelian differential with double poles at $x = \pm 1$ of the form
$$d\mathcal{Q} \underset{x\to\pm 1}{\sim} i dS_\pm.$$
Notice that the only (σ,τ)-dependence of the matrix $\Theta(x,\sigma,\tau)$ defined below enters through the b-periods of $d\mathcal{Q}(\sigma,\tau)$ in the expression
$$\boldsymbol{\zeta}_{\gamma(\sigma,\tau)} = \boldsymbol{\zeta}_{\gamma(0,0)} - \int_{\boldsymbol{b}} d\mathcal{Q}(\sigma,\tau),$$

and that the quantity entering in the exponents of expression (5.25) for $j_\pm$ is just (minus) the $\mathcal{B}_{g+1}$-period of $d\mathcal{Q}(\sigma,\tau)$, namely

$$\int_{\infty^-}^{\infty^+} d\mathcal{Q}(\sigma,\tau) = -\int_{\mathcal{B}_{g+1}} d\mathcal{Q}(\sigma,\tau).$$

- The reconstruction formulae (5.13) for the matrix $\Psi(x,\sigma,\tau) = (\boldsymbol{\psi}(x^+,\sigma,\tau), \boldsymbol{\psi}(x^-,\sigma,\tau))$ can be conveniently factored as follows

$$\Psi(x,\sigma,\tau) = e^{\frac{i}{2}\int_{\infty^-}^{\infty^+} d\mathcal{Q}(\sigma,\tau)} C(\sigma,\tau)\Theta(x,\sigma,\tau) e^{\Omega_S(x,\sigma,\tau)}, \tag{5.26}$$

where $C(\sigma,\tau)$ and $\Omega_S(x,\sigma,\tau)$ are diagonal matrices defined as

$$C(\sigma,\tau) = e^{-\frac{i}{2}\int_{\infty^-}^{\infty^+} d\mathcal{Q}(\sigma,\tau)\sigma_3}, \quad \Omega_S(x,\sigma,\tau) = \text{diag}\left(i\int_{\infty^+}^{x^+} d\mathcal{Q}(\sigma,\tau), i\int_{\infty^+}^{x^-} d\mathcal{Q}(\sigma,\tau)\right),$$

and the matrix $\Theta(x,\sigma,\tau)$ of θ-functions is defined by

$$\Theta(x,\sigma,\tau) = \\ \begin{pmatrix} k_-(x^+)\frac{\theta\left(\boldsymbol{\mathcal{A}}(x^+)-\boldsymbol{\zeta}_{\gamma(\sigma,\tau)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^+)-\boldsymbol{\zeta}_{\gamma(0,0)}\right)}{\theta\left(\boldsymbol{\mathcal{A}}(x^+)-\boldsymbol{\zeta}_{\gamma(0,0)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^+)-\boldsymbol{\zeta}_{\gamma(\sigma,\tau)}\right)} & k_-(x^-)\frac{\theta\left(\boldsymbol{\mathcal{A}}(x^-)-\boldsymbol{\zeta}_{\gamma(\sigma,\tau)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^+)-\boldsymbol{\zeta}_{\gamma(0,0)}\right)}{\theta\left(\boldsymbol{\mathcal{A}}(x^-)-\boldsymbol{\zeta}_{\gamma(0,0)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^+)-\boldsymbol{\zeta}_{\gamma(\sigma,\tau)}\right)} \\ k_+(x^+)\frac{\theta\left(\boldsymbol{\mathcal{A}}(x^+)-\boldsymbol{\zeta}_{\gamma'(\sigma,\tau)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^-)-\boldsymbol{\zeta}_{\gamma'(0,0)}\right)}{\theta\left(\boldsymbol{\mathcal{A}}(x^+)-\boldsymbol{\zeta}_{\gamma'(0,0)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^-)-\boldsymbol{\zeta}_{\gamma'(\sigma,\tau)}\right)} & k_+(x^-)\frac{\theta\left(\boldsymbol{\mathcal{A}}(x^-)-\boldsymbol{\zeta}_{\gamma'(\sigma,\tau)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^-)-\boldsymbol{\zeta}_{\gamma'(0,0)}\right)}{\theta\left(\boldsymbol{\mathcal{A}}(x^-)-\boldsymbol{\zeta}_{\gamma'(0,0)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^-)-\boldsymbol{\zeta}_{\gamma'(\sigma,\tau)}\right)} \end{pmatrix}.$$

Substituting (5.26) into (5.24) yields the required expression for $j_\pm$ expressed in terms of θ-functions through $\Theta(\pm 1,\sigma,\tau)$.

We can now also reconstruct the monodromy matrix corresponding to the reconstructed current (5.24) that would be obtained by substituting (5.24) directly into the definition (3.4) of $\Omega(x)$. For this we note that the monodromy matrix satisfies the differential equation (3.6) with the Lax connection now being given by (5.22), i.e.

$$\left[\partial_\pm - (\partial_\pm \Psi(x))\Psi(x)^{-1}, \Omega(x)\right] = 0.$$

This equation is solved by $\Omega(x) = \Psi(x)\Lambda(x)\Psi(x)^{-1}$ where $\Lambda(x)$ is a constant diagonal matrix, $\partial_\pm \Lambda(x) = 0$. The requirement that $\Omega(x)$ be the monodromy matrix fixes these integration constants to be the eigenvalues of the monodromy matrix, namely

$$\Lambda(x) = \text{diag}\,(e^{ip(x)}, e^{-ip(x)}),$$

so that the monodromy matrix of the reconstructed current (5.24) reads

$$\Omega(x) = \Psi(x)\begin{pmatrix} e^{ip(x)} & 0 \\ 0 & e^{-ip(x)} \end{pmatrix}\Psi(x)^{-1},$$

where $\Psi(x) = (\boldsymbol{\psi}(x^+), \boldsymbol{\psi}(x^-))$ and $\boldsymbol{\psi}$ is the reconstructed Baker-Akhiezer vector in (5.13).

The original field g of the SU(2) principal chiral model defined in (2.2) can also be recovered now that the current $j = -g^{-1}dg$ is known. It is the unique solution to

$$dg + gj = 0,$$

up to left multiplication by an arbitrary constant $U_L \in SL(2,\mathbb{C})$ matrix. In particular it is given by

$$g(\sigma,\tau) = U_L \cdot P\overleftarrow{\exp} \int_{(\sigma,\tau)} j, \tag{5.27}$$

where the integral runs from (σ,τ) on the worldsheet to an arbitrary but fixed point. For the moment though, before having imposed any reality conditions, this field takes values in $SL(2,\mathbb{C})$.

We conclude this section with a summary of the results obtained so far. In section 5.1 we constructed a map (5.8) associating to an equivalence class $[J(x,\sigma,\tau)]$ a set of algebro-geometric data $\{\Sigma,\ dp,\gamma(0,0)\} \in \mathcal{M}_{\mathbb{C}}^{(2g)}$. This map is injective, essentially by the Riemann-Roch theorem, and thus admits a (left) inverse (5.11) which is also injective. For any class $[J(x,\sigma,\tau)]$, only certain particular representatives, determined by the requirement $J(\infty) = 0$, are genuine solutions of the equations of motion. As was shown above, the condition $J(\infty) = 0$ picks out a family of solutions related by residual gauge transformations consisting of constant diagonal matrices $\tilde{g}$. This constant diagonal residual degree of freedom, corresponding to the $U(1)_R$ symmetric group preserving the 'highest weight' condition, was parametrised by introducing a new complex variable $W = \exp(i\bar{\theta}_0) \in \mathbb{C}^*$. So amending the algebro-geometric data $\mathcal{M}_{\mathbb{C}}^{(2g)}$ with this extra piece of data to form $\mathcal{M}_{\mathbb{C}} = \mathcal{M}_{\mathbb{C}}^{(2g)} \times \mathbb{C}^*$ we can describe the reconstruction of the current (5.25) by an injective map, which we call the *geometric map*,

$$\mathcal{G} : \mathcal{M}_{\mathbb{C}} \to \mathcal{S}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\infty}, \tag{5.28}$$

from the *moduli space* $\mathcal{M}_{\mathbb{C}} = \mathcal{M}_{\mathbb{C}}^{(2g)} \times \mathbb{C}^*$ describing the extended algebro-geometric data $\{\Sigma,\ dp,\gamma(0,0),\bar{\theta}_0\}$ into the space $\mathcal{S}_{\mathbb{C}}$ of complexified solutions $j \in \mathfrak{sl}(2,\mathbb{C})$ of the equations of motion (2.4) and the Virasoro constraints (2.6) (after imposing the final Virasoro constraint $\kappa_+ = \kappa_-$), which is a subspace of the space $\mathcal{M}_{\mathbb{C}}^{\infty}$ of all pairs $j = (j_0(\sigma,\tau), j_1(\sigma,\tau))$. The main feature of the geometric map specified by (5.25) which we wish to emphasise is that it linearises the motion on the moduli space $\mathcal{M}_{\mathbb{C}}$. In other words, once the initial data has been specified as a point on $\mathcal{M}_{\mathbb{C}}$, the subsequent evolution of the system is described by a linear flow on the corresponding $J(\Sigma)$-fibre of $\mathcal{M}_{\mathbb{C}}^{(2g)}$ as well as on the extra $\mathbb{C}^*$ factor of $\mathcal{M}_{\mathbb{C}}$

$$\begin{aligned} \boldsymbol{\theta}(\sigma,\tau) &= \boldsymbol{\theta}(0,0) - \boldsymbol{k}\sigma - \boldsymbol{w}\tau, \\ \bar{\theta}(\sigma,\tau) &= \bar{\theta}_0 - k_{\bar{\theta}}\sigma - w_{\bar{\theta}}\tau, \end{aligned} \tag{5.29}$$

where $\boldsymbol{\theta}(\sigma,\tau) = \boldsymbol{\mathcal{A}}(\gamma(\sigma,\tau)) + \boldsymbol{\theta}_0$ and $W = \exp(i\bar{\theta})$ is the coordinate along the $\mathbb{C}^*$ factor of $\mathcal{M}_{\mathbb{C}}$. The vector $\boldsymbol{\theta}_0 \in \mathbb{C}^g$ is a constant vector whose specific value will be chosen in section 5.5 to make the reality conditions on $\boldsymbol{\theta}$ simple. The velocities have been identified in the remarks following (5.25) as being linear combinations of all $K = g+1$ of the $\mathcal{B}$-periods of the spectral curve Σ defined in section 4, specifically

$$\boldsymbol{k} = \frac{1}{2\pi}\int_{\boldsymbol{b}} dp, \qquad \boldsymbol{w} = \frac{1}{2\pi}\int_{\boldsymbol{b}} dq,$$
$$k_{\bar\theta} = -\frac{1}{2\pi}\int_{\mathcal{B}_{g+1}} dp, \qquad w_{\bar\theta} = -\frac{1}{2\pi}\int_{\mathcal{B}_{g+1}} dq.$$

The first equation in (5.29) is a rewriting of (5.18) and the second equation for $\bar\theta$ comes from observing the reconstruction formula (5.25) written explicitly in terms of all the data.

5.3 The dual linear system

In section 5.5 we will show that the reconstructed current (5.24) corresponding to real algebro-geometric data is $\mathfrak{su}(2)$ valued, but in order to do so we will need the concept of the dual Baker-Akhiezer vector which we now introduce. We will derive a useful formula for the inverse matrix $\Psi(x,\sigma,\tau)^{-1}$ appearing in most of the reconstruction formulae, in particular (5.24), expressing it as a matrix of row vectors $\left(\boldsymbol{\psi}^+(x^+)^{\mathrm{T}}, \boldsymbol{\psi}^+(x^-)^{\mathrm{T}}\right)^{\mathrm{T}}$ where the row-vector $\boldsymbol{\psi}^+$ is the *dual Baker-Akhiezer vector* obeying

$$\boldsymbol{\psi}^+(x^\pm)\cdot\boldsymbol{\psi}(x^\pm) = 1, \quad \boldsymbol{\psi}^+(x^\pm)\cdot\boldsymbol{\psi}(x^\mp) = 0, \tag{5.30}$$

which can be constructed as follows [19]: consider a meromorphic differential $\widetilde\Omega$ with double poles at $\infty^\pm$ and $g+1$ specified zeroes at $\hat\gamma$. The set of $g+1$ remaining zeroes of $\widetilde\Omega$ define the dual divisor $\hat\gamma^+$ up to equivalence; indeed the differential $\widetilde\Omega$ is not unique, but the divisors $\hat\gamma^+, \hat\gamma'^+$ defined from two such differentials $\widetilde\Omega, \widetilde\Omega'$ are equivalent $\hat\gamma^+ \sim \hat\gamma'^+$. Then the dual Baker-Akhiezer vector $\boldsymbol{\psi}^+$ is defined as

$$\psi_i^+(P) = \chi(P)\widetilde\psi_i^+(P), \quad \text{with } \chi(P) = \frac{\widetilde\Omega(P)}{dx},$$

where $\widetilde\Omega$ is normalised by $\chi(\infty^+) = 1$ say, and where the components of the vector $\widetilde{\boldsymbol{\psi}}^+$ are Baker-Akhiezer functions specified by their respective divisors

$$(\widetilde\psi_1^+) \geq \left(\hat\gamma^+\right)^{-1}\infty^-, \quad (\widetilde\psi_2^+) \geq \left(\hat\gamma^+\right)^{-1}\infty^+,$$

as well as singular parts this time of opposite sign $-S_\pm$ at $x = \pm 1$ and are normalised such that

$$\psi_1^+(\infty^+) = \psi_2^+(\infty^-) = 1. \tag{5.31}$$

So just as the components of $\boldsymbol{\psi}$ were constructed in (5.13), those of $\widetilde{\boldsymbol{\psi}}^+$ can be constructed explicitly using a reconstruction formula analogous to (5.13) but with the replacements $\hat\gamma \to \hat\gamma^+$, $k_\pm \to h_\pm$ where $(h_-) = \gamma^+(0,0)\infty^-(\hat\gamma^+)^{-1}$ and $(h_+) = \gamma'^+(0,0)\infty^+(\hat\gamma^+)^{-1}$, as well as $dp \to -dp$, $dq \to -dq$ (and hence also $\boldsymbol{k} \to -\boldsymbol{k}$, $\boldsymbol{w} \to -\boldsymbol{w}$), namely we have

$$\widetilde\psi_1^+(P,\sigma,\tau) = h_-(P)\frac{\theta\left(\boldsymbol{\mathcal{A}}(P) - \boldsymbol{k}\sigma - \boldsymbol{w}\tau - \boldsymbol{\zeta}_{\gamma^+(0,0)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^+) - \boldsymbol{\zeta}_{\gamma^+(0,0)}\right)}{\theta\left(\boldsymbol{\mathcal{A}}(P) - \boldsymbol{\zeta}_{\gamma^+(0,0)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^+) - \boldsymbol{k}\sigma - \boldsymbol{w}\tau - \boldsymbol{\zeta}_{\gamma^+(0,0)}\right)}$$
$$\times \exp\left(-\frac{i\sigma}{2\pi}\int_{\infty^+}^P dp - \frac{i\tau}{2\pi}\int_{\infty^+}^P dq\right), \tag{5.32a}$$

$$\widetilde{\psi}_2^+(P,\sigma,\tau) = h_+(P)\frac{\theta\left(\boldsymbol{\mathcal{A}}(P) - \boldsymbol{k}\sigma - \boldsymbol{w}\tau - \boldsymbol{\zeta}_{\gamma'^+(0,0)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^-) - \boldsymbol{\zeta}_{\gamma'^+(0,0)}\right)}{\theta\left(\boldsymbol{\mathcal{A}}(P) - \boldsymbol{\zeta}_{\gamma'^+(0,0)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^-) - \boldsymbol{k}\sigma - \boldsymbol{w}\tau - \boldsymbol{\zeta}_{\gamma'^+(0,0)}\right)}$$
$$\times \exp\left(-\frac{i\sigma}{2\pi}\int_{\infty^-}^P dp - \frac{i\tau}{2\pi}\int_{\infty^-}^P dq\right), \quad (5.32b)$$

where $\gamma'^+(0,0)\infty^+ \sim \gamma^+(0,0)\infty^- \sim \hat{\gamma}^+$. That the vector $\boldsymbol{\psi}^+$ thus constructed indeed satisfies the orthogonality condition (5.30) and hence can be used to express the inverse matrix $\Psi(x,\sigma,\tau)^{-1} = \left(\boldsymbol{\psi}^+(x^+)^{\mathrm{T}}, \boldsymbol{\psi}^+(x^-)^{\mathrm{T}}\right)^{\mathrm{T}}$ is shown for example in [19]. Using the orthogonality relation (5.30) it is straightforward to show that $\boldsymbol{\psi}^+(P)$ is a left row eigenvector of the monodromy matrix $\Omega(x,\sigma,\tau)$ with the same eigenvalue as the right column eigenvector $\boldsymbol{\psi}(P)$ and furthermore that it is a solution to the *dual auxiliary linear system*, namely (c.f. the auxiliary linear system (5.2))

$$d\boldsymbol{\psi}^+ + \boldsymbol{\psi}^+ J(x,\sigma,\tau) = 0.$$

Defining the *dual dynamical divisor* $\gamma^+(\sigma,\tau)$ such that $\gamma^+(\sigma,\tau)\infty^-$ is the divisor of zeroes of $\widetilde{\psi}_1^+$, then for the same reasons that lead to (5.18), it follows from the analogue (5.32a) of equation (5.13a) for the first component of the un-normalised dual Baker-Akhiezer vector $\widetilde{\boldsymbol{\psi}}^+$ that the motion of the dual divisor is also linearised on $J(\Sigma)$ by the Abel map

$$\boldsymbol{\mathcal{A}}(\gamma^+(\sigma,\tau)) - \boldsymbol{\mathcal{A}}(\gamma^+(0,0)) = \boldsymbol{k}\sigma + \boldsymbol{w}\tau, \quad (5.33)$$

but the motion is in the opposite direction to that of the dynamical divisor $\gamma(\sigma,\tau)$ in (5.18).

5.4 Singular points and the dynamical divisor

Following a general procedure developed by Sklyanin ([24] and references therein) we show that the points of the dynamical divisor $\gamma(\sigma,\tau)$ along with the set of singular points $\{x_k\}$ of Γ can be conveniently characterised as the zeroes of the particular component $\mathcal{B}(x)$ of the monodromy matrix

$$\Omega(x) = \begin{pmatrix} \mathcal{A}(x) & \mathcal{B}(x) \\ \mathcal{C}(x) & \mathcal{D}(x) \end{pmatrix}. \quad (5.34)$$

To see this note that provided x is not a branch point, so that both eigenvectors are linearly independent, the monodromy matrix can be diagonalised by its matrix of (normalised) eigenvectors[12]

$$\Omega(x) = \begin{pmatrix} 1 & 1 \\ h_2(x^+) & h_2(x^-) \end{pmatrix}\begin{pmatrix} \Lambda(x^+) & 0 \\ 0 & \Lambda(x^-) \end{pmatrix}\begin{pmatrix} 1 & 1 \\ h_2(x^+) & h_2(x^-) \end{pmatrix}^{-1} \quad (5.35)$$

where $\Lambda(Q) = e^{ip(Q)} = e^{i\int_{\infty^+}^Q dp}$ is the function on Σ which at x^+, x^- gives the two eigenvalues $e^{ip(x)}, e^{-ip(x)}$ of $\Omega(x)$ since

$$\Lambda(x^+) = e^{i\int_{\infty^+}^{x^+} dp} = e^{ip(x)},$$

[12]This is simply a rewriting of equation (3.7), with $u(x,\sigma,\tau) = D(x,\sigma,\tau)\begin{pmatrix} 1 & 1 \\ h_2(x^+) & h_2(x^-) \end{pmatrix}^{-1}$ where $D(x,\sigma,\tau)$ is a diagonal matrix whose effect is simply to change the norms of the individual eigenvectors.

$$\Lambda(x^-) = e^{i\int_{\infty^+}^{x^-} dp} = e^{-i\int_{\infty^+}^{x^+} dp} = e^{-ip(x)},$$

where we have used the relation $\int_{\infty^+}^{x^+} dp + \int_{\infty^+}^{\hat{\sigma}x^+} dp \in 2\pi\mathbb{Z}$ along with $\hat{\sigma}x^+ = x^-$. Now because the function $h_2(P)$ is meromorphic on the algebraic curve $\Sigma : y^2 = \prod_{i=1}^{2g+2}(x - x_i)$ where the $\{x_i\}_{i=1}^{2g+2} = \{u_I, v_I\}_{I=1}^{g+1}$ denote the branch points, it can be written as a rational function in x and y. Its pole structure implies the following form

$$h_2(x^\pm) = \frac{\pm y_+ + h(x)}{\tilde{B}(x)},$$

where $\tilde{B}(x) = C\prod_{i=1}^{g}(x - x_{\gamma_i})$ with $x_{\gamma_i} = \hat{\pi}(\gamma_i)$ and $C = C(\sigma,\tau)$, $y_+ = y(x^+)$ is the value of y on the physical sheet and $h(x)$ is a polynomial of degree $g+1$.

The function $\mathcal{B}(x)$ can now be found from (5.35) to be

$$\mathcal{B}(x) = \left(\Lambda(x^+) - \Lambda(x^-)\right)\frac{\tilde{B}(x)}{2y_+}.$$

But the function $\Delta(x) = \left(\Lambda(x^+) - \Lambda(x^-)\right)^2 = \left(e^{ip(x)} - e^{-ip(x)}\right)^2$ is the discriminant of the quadratic polynomial $\Gamma(x,y)$ in y which has simple zeroes at the branch points, double zeroes at the singular points of the curve Γ and no other zeroes. Thus we see that the function $\mathcal{B}(x)$ has simple zeroes at the dynamical divisor and at the singular points of Γ. We shall therefore denote the singular points more naturally by $\gamma_i, i = -\infty, \ldots, 0$. The relation between singular points and points of the dynamical divisor is even stronger, as we show in appendix E, since the singular points can be thought of as 'trapped points' of the dynamical divisor in the sense that if we increase the genus of the curve Σ by blowing up certain singular points into genuine handles, then the number of points of the dynamical divisor increases by the same amount, as if the singular points in question had become dynamical.

When the zero x_{γ_i} of $\mathcal{B}(x)$ corresponds to a point $\gamma_i \in \Sigma$ of the dynamical divisor then using equation (5.35) we easily obtain the following asymptotics for the monodromy matrix (note that $\Lambda(P^+) = \Lambda(P^-)^{-1}$)

$$\Omega(x) \underset{x\to x_{\gamma_i}}{\longrightarrow} \begin{pmatrix} \Lambda(\gamma_i)^{-1} & 0 \\ \star & \Lambda(\gamma_i) \end{pmatrix},$$

from which it follows that $\mathcal{A}(x_{\gamma_i}) = \Lambda(\gamma_i)^{-1}$. This result also holds when the zero x_{γ_i} of $\mathcal{B}(x)$ corresponds to a singular point of Γ since in this case we have $\mathcal{A}(x_{\gamma_i}) = \Lambda(\gamma_i)^{-1} = \Lambda(\gamma_i)$.

Therefore the set of points $\gamma_i \in \Gamma, i = -\infty, \ldots, g$ comprising of the points $\gamma_i, i = 1, \ldots, g$ of the dynamical divisor and the singular points $\gamma_i, i = -\infty, \ldots, 0$ is uniquely characterised by the following conditions in terms of the two components $\mathcal{A}(x)$ and $\mathcal{B}(x)$ of $\Omega(x)$

$$\mathcal{B}(x_{\gamma_i}) = 0, \qquad \Lambda(\gamma_i)^{-1} = \mathcal{A}(x_{\gamma_i}). \tag{5.36}$$

By the exact same reasoning one can also argue from

$$\Omega(x) = \begin{pmatrix} 1 & k_2(x^+) \\ 1 & k_2(x^-) \end{pmatrix}^{-1} \begin{pmatrix} \Lambda(x^+) & 0 \\ 0 & \Lambda(x^-) \end{pmatrix} \begin{pmatrix} 1 & k_2(x^+) \\ 1 & k_2(x^-) \end{pmatrix},$$

where $k_2(P) = \psi_2^+(P)/\psi_1^+(P)$ has divisor $(k_2) \geq (\gamma^+(\sigma,\tau)\infty^-)^{-1}\infty^+$, that the set of points $\gamma_i^+, i = 1, \ldots, g$ of the dual divisor $\gamma^+(\sigma,\tau)$ along with the infinite set of singular points $\gamma_i^+ \equiv \gamma_i, i = -\infty, \ldots, 0$ of Γ can be uniquely characterised by

$$\mathcal{C}(x_{\gamma_i^+}) = 0, \qquad \Lambda(\gamma_i^+) = \mathcal{D}(x_{\gamma_i^+}). \tag{5.37}$$

Besides restricting the moduli of the algebraic curve, the reality condition (3.16) also imposes constraints on the dynamical divisor $\gamma(\sigma,\tau)$. The reality condition (3.16) on $\Omega(x)$ reads in components

$$\mathcal{D}(\bar{x}) = \overline{\mathcal{A}(x)}, \quad \mathcal{C}(\bar{x}) = -\overline{\mathcal{B}(x)}.$$

Using these relations the equation (5.36) characterising the dynamical divisor (and singular points) can be rewritten as

$$\mathcal{C}(x_{\hat{\tau}\gamma_i}) = 0, \qquad \Lambda(\hat{\tau}\gamma_i) = \mathcal{D}(x_{\hat{\tau}\gamma_i}), \quad i = -\infty, \ldots, g$$

where we have used the fact that the anti-holomorphic involution $\hat{\tau}$ defined in (4.19) maps the point $\gamma_i = (x_{\gamma_i}, \Lambda(\gamma_i))$ to $\hat{\tau}\gamma_i = (x_{\hat{\tau}\gamma_i}, \Lambda(\hat{\tau}\gamma_i)) = \left(\bar{x}_{\gamma_i}, \overline{\Lambda(\gamma_i)}^{-1}\right)$. But this last equation expresses the fact that the image $\hat{\tau}\gamma(\sigma,\tau)$ of the dynamical divisor along with the images $\hat{\tau}\gamma_i, i = -\infty, \ldots, 0$ of the singular points are precisely the dual dynamical divisor and the singular points themselves by comparison with (5.37), i.e.

$$\hat{\tau}\gamma(\sigma,\tau) = \gamma^+(\sigma,\tau), \qquad \hat{\tau}\gamma_i = \gamma_i, \quad i = -\infty, \ldots, 0. \tag{5.38}$$

Thus we conclude that the singular points all lie along the real axis, which as we already know accumulate at $x = \pm 1$, and the dual dynamical divisor $\gamma^+(\sigma,\tau)$ is nothing but the reflection $\hat{\tau}\gamma(\sigma,\tau)$ of the dynamical divisor $\gamma(\sigma,\tau)$ through the real axis.

5.5 Reality conditions

We can describe the reality conditions (5.38) on the dynamical divisor more explicitly. By definition of the dual dynamical divisor we know that $\gamma(0,0) \cdot \gamma^+(0,0) \sim Z \cdot (\infty^+)^2$ where $Z = (\widetilde{\Omega})$ is the canonical class and so

$$\gamma(\sigma,\tau) \cdot (\hat{\tau}\gamma(\sigma,\tau)) \sim \gamma(0,0) \cdot (\hat{\tau}\gamma(0,0)) \sim \hat{\gamma} \cdot (\hat{\tau}\hat{\gamma}) \cdot (\infty^-)^{-2} \sim Z \cdot (\infty^+)^2,$$

where the first equivalence follows from the fact that $k_-^{-1} h_-^{-1} \psi_1 \widetilde{\psi}_1^+$ is meromorphic or equivalently from equations (5.18) and (5.33) which together imply

$$\boldsymbol{\mathcal{A}}(\gamma(\sigma,\tau)) + \boldsymbol{\mathcal{A}}(\hat{\tau}\gamma(\sigma,\tau)) = \boldsymbol{\mathcal{A}}(\gamma(0,0)) + \boldsymbol{\mathcal{A}}(\hat{\tau}\gamma(0,0)), \tag{5.39}$$

and then invoking Abel's theorem. Now to obtain the induced action of the anti-holomorphic involution $\hat{\tau}$ on the Jacobian $J(\Sigma)$ consider a positive divisor $D = P_1 \ldots P_g$ of degree $\deg D = g$, then

$$\boldsymbol{\mathcal{A}}(\hat{\tau}D) = 2\pi \sum_{i=1}^{g} \int_{\infty^+}^{\hat{\tau}P_i} \boldsymbol{\omega} = 2\pi \sum_{i=1}^{g} \int_{\infty^+}^{P_i} \hat{\tau}^* \boldsymbol{\omega} = -2\pi \sum_{i=1}^{g} \int_{\infty^+}^{P_i} \overline{\boldsymbol{\omega}} = -\overline{\boldsymbol{\mathcal{A}}(D)}.$$

It follows then from (5.39) that the dynamical divisor $\gamma(\sigma,\tau)$ corresponding to real solutions evolves such that

$$2\,\mathrm{Im}\,\boldsymbol{\mathcal{A}}(\gamma(\sigma,\tau)) = 2\,\mathrm{Im}\,\boldsymbol{\mathcal{A}}(\gamma(0,0)) = \boldsymbol{\mathcal{A}}(Z\cdot(\infty^+)^2).$$

In other words, the dynamical divisor $\gamma(\sigma,\tau)$ corresponding to real solutions is constrained to move on the real g-dimensional sub-torus T^g of the complex g-dimensional Jacobian $J(\Sigma)$ defined by

$$T^g = \left\{\boldsymbol{X}\in J(\Sigma) \quad | \quad 2\,\mathrm{Im}\,\boldsymbol{X} = \boldsymbol{\mathcal{A}}(Z\cdot(\infty^+)^2)\right\} \subset J(\Sigma). \tag{5.40}$$

The reality of the singular parts at $x=\pm 1$ of both dp and dq in (4.9) and (5.16) following from the reality condition on the quasi-momentum (3.17) implies that (using also the fact that the local parameter x is such that $\hat{\tau}^* x = \bar{x}$, by definition of $\hat{\tau}$)

$$\overline{\hat{\tau}^* dp} = dp, \qquad \overline{\hat{\tau}^* dq} = dq,$$

because a normalised (vanishing a-periods) Abelian differential is uniquely specified by its singular parts. It follows from this and (4.20) that $\boldsymbol{k},\boldsymbol{w}\in\mathbb{R}^g$ since

$$\begin{aligned}\overline{k_i} &= \frac{1}{2\pi}\int_{b_i}\overline{dp} = \frac{1}{2\pi}\int_{\hat{\tau}b_i}\overline{\hat{\tau}^* dp} = \frac{1}{2\pi}\int_{b_i} dp = k_i,\\ \overline{w_i} &= \frac{1}{2\pi}\int_{b_i}\overline{dq} = \frac{1}{2\pi}\int_{\hat{\tau}b_i}\overline{\hat{\tau}^* dq} = \frac{1}{2\pi}\int_{b_i} dq = w_i.\end{aligned} \tag{5.41}$$

Therefore we find again that once the initial divisor $\gamma(0,0)$, which is part of the algebro-geometric data, is chosen to live on T^g, the dynamical divisor $\gamma(\sigma,\tau)$ is constrained to move on the g-dimensional real sub-torus $T^g\subset J(\Sigma)$. Thus the restriction on the algebro-geometric data corresponding to real solutions can be summarised as

$$\boldsymbol{\mathcal{A}}\left(\gamma(0,0)\right)\in T^g. \tag{5.42}$$

Moreover, the reality condition $j_\pm^\dagger = -j_\pm$ has the effect of reducing the $\mathbb{C}^*$ action on the space of complexified solutions $\mathcal{S}_\mathbb{C}$ discussed at the end of section 5.2 to a $U(1)_R$ action on the space of real solutions $\mathcal{S}_\mathbb{R}$. This yields the simple reality condition $\bar{\theta}_0\in\mathbb{R}/2\pi\mathbb{Z}$ on the $\bar{\theta}_0$ component of the extended algebro-geometric data. Furthermore, the g-dimensional real sub-torus (5.42) of $J(\Sigma)$ can be parametrised by the g component vector $\boldsymbol{\theta}=(\theta_1,\dots,\theta_g)$ defined for every point $\boldsymbol{X}\in T^g$ by

$$\boldsymbol{\theta} = \boldsymbol{X} + \boldsymbol{\theta}_0,$$

which can be made *real* provided we choose $-\boldsymbol{\theta}_0\in T^g$, so we set for instance

$$\boldsymbol{\theta}_0 = -\frac{1}{2}\boldsymbol{\mathcal{A}}(Z\cdot(\infty^+)^2). \tag{5.43}$$

Through these coordinates the sub-torus $T^g\subset J(\Sigma)$ is identified with the real torus $\mathbb{R}^g/2\pi\mathbb{Z}^g$ which $\boldsymbol{\theta}$ takes values in.

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Conversely we now show that if the algebro-geomtric data is *real* so that the initial divisor satisfies $\hat{\tau}\gamma(0,0) = \gamma^+(0,0)$ then the current j given by the reconstruction formula (5.24) is real, i.e. $j^\dagger = -j$. Since $\hat{\tau}\gamma(0,0) = \gamma^+(0,0)$ we have $\hat{\tau}\hat{\gamma} \sim \hat{\gamma}^+$ and so we choose the form $\widetilde{\Omega}$ such that equality actually holds, namely $\hat{\tau}\hat{\gamma} = \hat{\gamma}^+$. Now consider the functions

$$f_i(P,\sigma,\tau) = \widetilde{\psi}_i^+(P,\sigma,\tau) \Big/ \overline{\psi_i(\hat{\tau}P,\sigma,\tau)}.$$

These are meromorphic functions with at most g poles and hence are constant by the Riemann-Roch theorem. These constants are all fixed by the normalisation conditions (5.31) to be equal to 1 since $f_i(\infty^+,\sigma,\tau) = 1$ and so we obtain the following reality condition on the Baker-Akhiezer vector

$$\boldsymbol{\psi}(\hat{\tau}P,\sigma,\tau)^\dagger = \widetilde{\boldsymbol{\psi}}^+(P,\sigma,\tau). \tag{5.44}$$

Now using the dual Baker-Akhiezer vector to construct the inverse matrix $\Psi(x,\sigma,\tau)^{-1}$ it can be written as follows

$$\Psi(x,\sigma,\tau)^{-1} = \left(\boldsymbol{\psi}^+(x^+)^{\mathrm{T}}, \boldsymbol{\psi}^+(x^-)^{\mathrm{T}}\right)^{\mathrm{T}} = \mathrm{diag}\left(\chi(x^+),\chi(x^-)\right)\left(\widetilde{\boldsymbol{\psi}}^+(x^+)^{\mathrm{T}}, \widetilde{\boldsymbol{\psi}}^+(x^-)^{\mathrm{T}}\right)^{\mathrm{T}},$$

where $\chi(P) = \frac{\widetilde{\Omega}(P)}{dx}$ is meromorphic with zeroes at $\hat{\gamma}$ and $\hat{\tau}\hat{\gamma}$ and poles at the $2g+2$ branch points which we denote by the divisor B, i.e.

$$(\chi) = \hat{\gamma}\cdot(\hat{\tau}\hat{\gamma})\cdot B^{-1}.$$

Using the reality condition on the algebraic curve Σ we can split the divisor B of branch points into two divisors B_+, B_- such that

$$B = B_+\cdot B_-, \quad B_+ = \hat{\tau}B_-.$$

Thus we can write the meromorphic function χ as a product $\chi = \chi_+\chi_-$ of two meromorphic functions $\chi_\pm$ with respective divisors

$$(\chi_+) = \hat{\gamma}\cdot B_+^{-1}, \quad (\chi_-) = \hat{\gamma}^+\cdot B_-^{-1},$$

and which are therefore related by $\chi_+(P) = \overline{\chi_-(\hat{\tau}P)}$. The reconstructed current (5.24) now takes the form

$$\begin{aligned} j_+(\sigma,\tau) &= i\kappa_+\left(\Psi_0\ \mathrm{diag}\left(\chi_+(1^+),\chi_+(1^-)\right)\sigma_3\ \mathrm{diag}\left(\chi_-(1^+),\chi_-(1^-)\right)\widetilde{\Psi}_0^+\right), \\ j_-(\sigma,\tau) &= i\kappa_-\left(\Phi_0\ \mathrm{diag}\left(\chi_+(1^+),\chi_+(1^-)\right)\sigma_3\ \mathrm{diag}\left(\chi_-(1^+),\chi_-(1^-)\right)\widetilde{\Phi}_0^+\right), \end{aligned} \tag{5.45}$$

where $\widetilde{\Psi}_0^+, \widetilde{\Phi}_0^+$ are the leading terms in the expansion of $\left(\widetilde{\boldsymbol{\psi}}^+(x^+)^{\mathrm{T}}, \widetilde{\boldsymbol{\psi}}^+(x^-)^{\mathrm{T}}\right)^{\mathrm{T}}$ near $x = \pm 1$ just as Ψ_0, Φ_0 were the leading terms for $\Psi(x,\sigma,\tau)$ in the expansion (5.19). Using the reality condition (5.44) on the Baker-Akiezer vector we find $\widetilde{\Psi}_0^+ = \Psi_0^\dagger, \widetilde{\Phi}_0^+ = \Phi_0^\dagger$ and since also

$$\mathrm{diag}\left(\chi_-(1^+),\chi_-(1^-)\right) = \mathrm{diag}\left(\chi_+(1^+),\chi_+(1^-)\right)^\dagger,$$

it follows from the reconstruction formulae written in the form (5.45) that the reconstructed currents are anti-hermitian $j_\pm^\dagger = -j_\pm$, i.e. $j_\pm \in \mathfrak{su}(2)$. Now with the extra reality condition $\bar{\theta}_0 \in \mathbb{R}/2\pi\mathbb{Z} = S^1$ on the extension of the algebro-geometric data, the reconstruction formula (5.24) is also easily checked to give an anti-hermitian current so that $j_\pm \in \mathfrak{su}(2)$. It therefore follows that as a result of imposing the reality conditions, the reconstructed field g in (5.27) becomes SU(2)-valued, and in particular, the original fields $X_1, \dots, X_4$ describing the embedding of the string into the $S^3 \subset \mathbb{R}^4$ part of the target space are real valued as required.

To summarise the discussion of reality conditions, the *real* algebro-geometric data which gives rise to real solutions through the geometric map (5.28) can be identified with a sub-bundle $\mathcal{M}_\mathbb{R} = \mathcal{M}_\mathbb{R}^{(2g)} \times S^1$ of the extended bundle $\mathcal{M}_\mathbb{C} = \mathcal{M}_\mathbb{C}^{(2g)} \times \mathbb{C}^*$ introduced in section 5.2, namely

$$T^g \to \mathcal{M}_\mathbb{R}^{(2g)} \to \mathcal{L}_\mathbb{R}\,, \tag{5.46}$$

where $\mathcal{L}_\mathbb{R}$ is the real part of the leaf $\mathcal{L}$ parametrised by real values of the filling fractions (4.13). The restriction $\mathcal{G}_\mathbb{R} = \mathcal{G}|_{\mathcal{M}_\mathbb{R}}$ of the geometric map (5.28) to the real bundle $\mathcal{M}_\mathbb{R} \subset \mathcal{M}_\mathbb{C}$ is an injective map from *real* algebro-geometric data to *real* solutions in $\mathcal{S}_\mathbb{R}$,

$$\begin{tikzcd}
\mathcal{M}_\mathbb{C} \arrow[r, "\mathcal{G}"] & \mathcal{S}_\mathbb{C} \subset \mathcal{M}_\mathbb{C}^\infty \\
\mathcal{M}_\mathbb{R} \arrow[r, "\mathcal{G}_\mathbb{R}"] \arrow[u, hook] & \mathcal{S}_\mathbb{R} \subset \mathcal{M}_\mathbb{R}^\infty \arrow[u, hook]
\end{tikzcd}$$

5.6 Periodicity conditions

Since the configuration of a finite-gap solution is specified by the position of the point $\mathcal{A}(\gamma(\sigma,\tau)) \in J(\Sigma)$ on the Jacobian of Σ, a necessary condition for the solution to be periodic is that the motion of this point be periodic on $J(\Sigma)$. But by (5.18) we know that $\mathcal{A}(\gamma(\sigma,\tau))$ moves linearly on $J(\Sigma)$ in both σ and τ so that periodic motion under $\sigma \to \sigma + 2\pi$ or $\tau \to \tau + T$ can occur respectively only when the following conditions are met

$$\begin{aligned} 2\pi \boldsymbol{k} &= 2\pi \boldsymbol{n} + 2\pi \Pi \boldsymbol{m}, \qquad && \boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}^g, \\ T\boldsymbol{w} &= 2\pi \boldsymbol{n}' + 2\pi \Pi \boldsymbol{m}', \qquad && \boldsymbol{n}', \boldsymbol{m}' \in \mathbb{Z}^g. \end{aligned} \tag{5.47}$$

Combining these periodicity conditions with the reality conditions (5.41) on the vectors $\boldsymbol{k}$ and $\boldsymbol{w}$ as well as the fact that the components of Π are pure imaginary we see that the motion is periodic under $\sigma \to \sigma + 2\pi$ and $\tau \to \tau + T$ respectively when

$$\begin{aligned} \boldsymbol{k} &\in \mathbb{Z}^g, \\ \frac{T}{2\pi} \boldsymbol{w} &\in \mathbb{Z}^g. \end{aligned} \tag{5.48}$$

Note that the σ-periodicity condition is precisely equivalent to the integrality of the b-period of the differential dp in (4.6).

However these conditions are not sufficient. If we look at the explicit reconstruction formula (5.25) for $j_\pm$ we find that the periodicity in $\sigma \to \sigma + 2\pi$ and $\tau \to \tau + T$ of the similarity transformation corresponding to the S^1 factor of the algebro-geometric data $\mathcal{M}_\mathbb{R}$ requires also that the $\mathcal{B}_{g+1}$-period of the differentials dp and dq be integer valued respectively,

$$\begin{aligned} k_{\bar{\theta}} &= \frac{1}{2\pi} \int_{\infty^-}^{\infty^+} dp \in \mathbb{Z}, \\ T w_{\bar{\theta}} &= \frac{T}{2\pi} \int_{\infty^-}^{\infty^+} dq \in \mathbb{Z}. \end{aligned} \tag{5.49}$$

To show that the conditions (5.48) and (5.49) are sufficient to ensure periodicity of the reconstructed $j(\sigma, \tau)$ we note from (5.13) and using the property (5.14) of θ-functions that

$$\begin{aligned} \boldsymbol{\psi}(P, \sigma + 2\pi, \tau) &= \exp\left\{ i \int_{\infty^+}^{P} dp \right\} \boldsymbol{\psi}(P, \sigma, \tau), \\ \boldsymbol{\psi}(P, \sigma, \tau + T) &= \exp\left\{ \frac{iT}{2\pi} \int_{\infty^+}^{P} dq \right\} \boldsymbol{\psi}(P, \sigma, \tau). \end{aligned}$$

Since the exponent in these expressions are (σ, τ)-independent it follows that the matrix $\Psi(x, \sigma, \tau) = (\boldsymbol{\psi}(P^+, \sigma, \tau), \boldsymbol{\psi}(P^-, \sigma, \tau))$ gets multiplied on the right by a (σ, τ)-independent diagonal matrix under $\sigma \to \sigma + 2\pi$ or $\tau \to \tau + T$, and hence (5.24) is indeed periodic

$$\begin{aligned} j_\pm(x, \sigma + 2\pi, \tau) &= j_\pm(x, \sigma, \tau), \\ j_\pm(x, \sigma, \tau + T) &= j_\pm(x, \sigma, \tau). \end{aligned}$$

But periodicity of the current $j = -g^{-1} dg$, although necessary, is not sufficient to ensure periodicity of the original field g. The extra condition necessary for periodicity of $g(\sigma, \tau)$ in σ can be obtained as follows. Starting from the formula (5.27) for the reconstruction of g

$$g(\sigma, \tau) = U_L \cdot P\overleftarrow{\exp} \int_{(\sigma,\tau)} j,$$

we compare this expression to the same expression translated by $\sigma \to \sigma + 2\pi$, namely

$$g(\sigma + 2\pi, \tau) = U_L \cdot P\overleftarrow{\exp} \int_{(\sigma+2\pi,\tau)} j.$$

Its inverse is given by $g^{-1}(\sigma + 2\pi, \tau) = \left(P\overleftarrow{\exp} \int^{(\sigma+2\pi,\tau)} j \right) \cdot U_L^{-1}$ so that

$$g^{-1}(\sigma + 2\pi, \tau) g(\sigma, \tau) = P\overleftarrow{\exp} \int_{(\sigma,\tau)}^{(\sigma+2\pi,\tau)} j = \Omega(0, \sigma, \tau).$$

The last equality comes from the definition of the monodromy matrix (3.4). Periodicity of $g(\sigma, \tau)$ under $\sigma \to \sigma + 2\pi$ is therefore guaranteed provided

$$\Omega(0, \sigma, \tau) = \mathbf{1}.$$

This condition however is equivalent to the condition that $p(0) = 2\pi m \in 2\pi\mathbb{Z}$ or stated as a condition on a certain period of dp, as for the other periodicity conditions (5.48) and (5.49)

$$\frac{1}{2\pi}\int_{\infty^+}^{0^+} dp \in \mathbb{Z}. \tag{5.50}$$

Therefore if the real geometric map $\mathcal{G}_{\mathbb{R}}$ is restricted to a sub-bundle of the algebro-geometric data $\mathcal{M}_{\mathbb{R}}$ corresponding to data satisfying the periodicity conditions (5.48), (5.49) and (5.50) then its image will consist of real periodic solutions and the corresponding reconstructed field $g(\sigma,\tau)$ will be SU(2)-valued and σ-periodic, as required.

5.7 String motion as rigid, linear motion on a torus

The reconstructed solutions described in the previous sections have the important property that all dependence on the worldsheet coordinates σ and τ is contained in the linear evolution (5.29) of the Jacobian coordinates $\boldsymbol{\theta}(\sigma,\tau)$ and of the $U(1)_R$ coordinate $\bar{\theta}(\sigma,\tau)$,

$$j(\sigma,\tau) = j[\boldsymbol{\theta}(\sigma,\tau), \bar{\theta}(\sigma,\tau)].$$

For real solutions, $\bar{\theta}$ and each component θ_i of $\boldsymbol{\theta}$ are real and have period 2π. Together these angular variables parametrise a torus $T^K = T^g \times S^1$ in $\mathcal{M}_{\mathbb{R}}$. It is convenient to consider a slightly different choice of basis cycles on T^K by defining new angular coordinates, $\vec{\varphi} = (\varphi_1, \dots, \varphi_K)$ via,

$$\varphi_I = \theta_i - \bar{\theta} \qquad \text{for } I = i = 1,\dots,g = K-1, \qquad \varphi_K = -\bar{\theta}. \tag{5.51}$$

In these variables, the linear evolution takes the very simple form

$$\varphi_I(\sigma,\tau) = \varphi_I(0,0) - n_I\sigma - v_I\tau \tag{5.52}$$

where,

$$\int_{\mathcal{B}_I} dp = 2\pi n_I, \qquad \int_{\mathcal{B}_I} dq = 2\pi v_I. \tag{5.53}$$

At fixed τ, the linear evolution (5.52) corresponds to a configuration of a wrapped string on the real torus T^K. Any finite gap solution is effectively described by a motion of this string on T^K which has the following features:

- The string wraps a non-contractible cycle on the torus. Representing T^K as $\mathbb{R}^K/2\pi \times \mathbb{Z}^K$ the cycle corresponds to the lattice vector $-\vec{n}$ where $\vec{n} = (n_1,\dots,n_K)$. In other words, the winding numbers of the string around the fundamental cycles on T^K are determined by the mode numbers n_I of the finite-gap solution.

- The resulting time evolution corresponds to rigid, linear motion of the wrapped string on T^K with constant velocity vector $-\vec{v}$ where $\vec{v} = (v_1,\dots,v_K)$.

- The filling fractions $\vec{\mathcal{S}} = (\mathcal{S}_1,\dots,\mathcal{S}_K)$ play the role of conjugate momenta to the string coordinates $\vec{\varphi} = (\varphi_1,\dots,\varphi_K)$ (this point will be made more precise in the next subsection). In particular, from (4.17), the momentum constraint $\mathcal{P} = 0$ implies $\vec{\mathcal{S}}\cdot\vec{n} = 0$ which means that the momentum carried by the string is perpendicular to the line on which it lies at fixed τ.

The features described above can be summarised by saying that the motion resembles that of an infinitely rigid string moving freely on the torus T^g with winding numbers determined by the mode numbers of the finite-gap solution and momenta determined by the filling fractions.

5.8 The Hamiltonian description

In this final subsection we will show that the σ, τ evolution of the string described above admits a simple Hamiltonian description in terms of a finite dimensional integrable system with the moduli space $\mathcal{M}_{\mathbb{R}}^{(2g)}$ as its phase space. As reviewed in Appendix B, the construction of [13] provides a natural symplectic form on $\mathcal{M}_{\mathbb{R}}^{(2g)}$. In terms of the coordinates $\{S_i, \theta_i\}$ this takes the form,

$$\omega_{2g} = \sum_{i=1}^{g} \delta S_i \wedge \delta\theta_i$$

where δ denotes the exterior derivative on the moduli space. The symplectic form defines a Poisson bracket for the coordinates,

$$\{S_i, S_j\} = \{\theta_i, \theta_j\} = 0, \quad \{S_i, \theta_j\} = \delta_{ij}.$$

With the above Poisson bracket any function of the conserved quantities $\{S_i\}$ generates a linear flow on the Jacobian. To define the appropriate evolution, we consider the functionals,

$$\begin{aligned}
\mathcal{P} &= -\frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \, \mathrm{tr} \left[j_0 j_1 \right], \\
\mathcal{E} &= -\frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \, \frac{1}{2} \mathrm{tr} \left[j_0^2 + j_1^2 \right].
\end{aligned}$$

These quantities are the Noether charges for rigid translations of the worldsheet coordinates σ and τ respectively. They are manifestly independent of σ and their independence of τ follows from the corresponding conservation laws for energy and momentum on the worldsheet. As $\mathcal{P}$ and $\mathcal{E}$ are uniquely determined by the time independent data (Σ, dp) they correspond to well-defined functions on the leaf $\mathcal{L}$ which forms the base of the fibration $\mathcal{M}_{\mathbb{R}}^{(2g)}$. Explicitly, we define the corresponding Hamiltonian functions,

$$\begin{aligned}
H_\sigma &= \mathcal{P}[S_1, \ldots, S_g], \\
H_\tau &= \mathcal{E}[S_1, \ldots, S_g].
\end{aligned}$$

We will now consider the linear flows on $\mathcal{M}_{\mathbb{R}}^{(2g)}$ generated by these functions with parameters σ and τ respectively. The corresponding Hamilton equations are,

$$\frac{\partial\theta_i}{\partial\sigma} = \frac{\partial\mathcal{P}}{\partial S_i}, \quad \frac{\partial\theta_i}{\partial\tau} = \frac{\partial\mathcal{E}}{\partial S_i}. \tag{5.54}$$

The variations of $\mathcal{P}$ and $\mathcal{E}$ along the leaf $\mathcal{L}$ were given in section 4.2 (see also Appendix C). In particular, the equations (4.15) imply,

$$\frac{\partial\mathcal{P}}{\partial S_i} = -k_i, \quad \frac{\partial\mathcal{E}}{\partial S_i} = -\omega_i,$$

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where, as before,

$$k_i = \frac{1}{2\pi}\int_{b_i} dp, \quad w_i = \frac{1}{2\pi}\int_{b_i} dq.$$

Integrating (5.54) we immediately obtain the linear flow,

$$\theta_i(\sigma,\tau) = \theta_i(0,0) - k_i\sigma - \omega_i\tau \tag{5.55}$$

which is equivalent to the first equality in (5.29).

To obtain the physical phase space of the model we must impose the remaining constraint $\mathcal{P} = 0$ and then identifying points related by translations of the worldsheet coordinate σ. The combined effect of these steps can be described as a symplectic reduction of the integrable system $(\mathcal{M}_{\mathbb{R}}^{(2g)}, \omega_{2g})$. Specifically we work with the moment map,

$$\mu = H_\sigma = -\sum_{I=1}^{K} n_I \mathcal{S}_I = -\sum_{i=1}^{g} k_i S_i - \frac{n_K}{2}(L-R)$$

where we have used (4.17). We impose the constraint $\mu = 0$ and at the same time gauge the corresponding linear flow,

$$X_\mu = \sum_{i=1}^{g} k_i \frac{\partial}{\partial\theta_i}$$

which defines a $U(1)$ action on $\mathcal{M}_{\mathbb{R}}^{(2g)}$. A reduced phase-space $\tilde{\mathcal{M}}_{\mathbb{R}}^{(2g-2)}$ with symplectic form $\tilde{\omega}_{2g-2}$ can then be defined using standard symplectic reduction,

$$\begin{tikzcd} \mu^{-1}(0) \arrow[r, hook, "i"] \arrow[d, "\pi"] & \mathcal{M}_{\mathbb{R}}^{(2g)} \\ \tilde{\mathcal{M}}_{\mathbb{R}}^{(2g-2)} & \end{tikzcd}$$

where i is the inclusion of $\mu^{-1}(0)$ in $\mathcal{M}_{\mathbb{R}}^{(2g)}$ and π is a projection onto the base of the corresponding principal $U(1)$ bundle. More precisely, the resulting quotient is well-defined everywhere except at the degenerate point where $S_i = 0$ for all i, which is a fixed point of the $U(1)$ action.

As mentioned in the Introduction, the striking feature of the Hamiltonian description given above, is that the canonically normalised action variables are the filling fractions S_i, defined in (4.12). A leading-order semiclassical treatment of the model would therefore yield a quantisation of the filling fractions in integer units.

So far we have only discussed the Hamiltonian evolution of the coordinates on the reduced moduli space $\mathcal{M}_{\mathbb{R}}^{(2g)}$ corresponding to linear motion of the Jacobian angles θ_i. However, the reconstructed solution also has additional dependence on σ and τ which originates to linear evolution of the $U(1)_R$ angle $\bar{\theta}$. To obtain a more complete Hamiltonian description we should consider an enlarged phase space with two additional dimensions. In particular, the extra coordinate $\bar{\theta}$ corresponds to a global $U(1)_R$ rotation and is therefore canonically conjugate to the conserved Noether charge R define above. Thus we should allow R to vary and include it as one of the moduli of the solution. In particular we should then consider a fibration whose base is the moduli space of admissible pairs (Σ, dp) *without* fixing the value of R. The corresponding leaf $\hat{\mathcal{L}}$ in the universal moduli space would have dimension $g+1$ and the resulting moduli space of dimension $2g+2$ would be a $J(\Sigma) \times \mathbb{C}^*$ fibration of $\hat{\mathcal{L}}$. Although we will not pursue this idea here we note in passing that the full σ, τ evolution of the finite gap solution as given in (5.52) can easily be reproduced by considering the Hamiltonians $\mathcal{P}$, $\mathcal{E}$ defined above as functions of the $K = g+1$ filling fractions $\mathcal{S}_I$ and working with the symplectic form,

$$\hat{\omega}_{2K} = \sum_{I=1}^{K} \delta \mathcal{S}_I \wedge \delta \varphi_I .$$

Finally we note that the additional σ, τ dependence of the solution originating from the angle $\bar{\theta}$ can be eliminated by working in an appropriate reference frame. Specifically, we can define new coordinates on the target $S^3 \simeq \mathrm{SU}(2)$ by setting $\tilde{g} = gh \in \mathrm{SU}(2)$ with,

$$h(\sigma, \tau) = e^{\left(\frac{i}{2}\bar{\theta}_0 - \frac{i}{2}\int_{\infty^-}^{\infty^+} d\mathcal{Q}\right)\sigma_3} \in \mathrm{SU}(2)$$

where, as above,

$$d\mathcal{Q}(\sigma, \tau) = \frac{1}{2\pi}\left(\sigma dp + \tau dq\right),$$

with a corresponding right-current,

$$\tilde{j} = -\tilde{g} d\tilde{g} = h^{-1} j h - h^{-1} dh$$

which satisfy the equations of motion,

$$d\tilde{j} - \tilde{j} \wedge \tilde{j} = 0, \quad \nabla * \tilde{j} = d * \tilde{j} - [h^{-1}dh, *\tilde{j}] = 0.$$

The new coordinates correspond to a frame which undergoes the same constant $\mathrm{SU}(2)_R$ rotation as the string itself at each point along its length. This is analogous to the body-fixed reference frame used in the standard analysis of a rigid rotation body. The particular merit of this frame is that, in these coordinates, the σ, τ-dependence of the solution is purely contained in the internal variables $\boldsymbol{\theta}$;

$$\tilde{j}(\sigma, \tau) = \tilde{j}[\boldsymbol{\theta}(\sigma, \tau)].$$

Thus, in this frame, the evolution of the finite-gap solution in the worldsheet coordinates is precisely equivalent to that of the integrable system on $\mathcal{M}_{\mathbb{R}}^{(2g)}$ introduced above.

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A. The desingularised curve

Since the matrix $L(x,\sigma,\tau)$ is 2×2 the equation for the curve $\widehat{\Sigma}$ can be written more explicitly as

$$\widehat{\Sigma}: \quad \widehat{\Sigma}(x,y) = P_2(x)y^2 + P_0(x) = 0, \tag{A.1}$$

where $P_0(x), P_2(x)$ are polynomials and the term linear in y is absent because the sum of the eigenvalues $\{p'(x), -p'(x)\}$ of $L(x,\sigma,\tau)$ is zero. The discriminant for such a polynomial $\widehat{\Sigma}(x,y)$ in y is given by

$$\Delta(x) = -4\frac{P_0(x)}{P_2(x)}. \tag{A.2}$$

It expresses the square of the difference between the roots of the quadratic polynomial $\widehat{\Sigma}(x,y)$ in y. Now as argued in [3, 4], to exclude unphysical branch points which arise from odd multiplicity zeroes of the discriminant (A.2) but retain the physical branch points which arise as simple poles of the discriminant (A.2) one should impose the condition

$$P_2(x)\Delta(x) = (Q(x))^2\,,$$

where $Q(x)$ is some polynomial. This implies that $-4P_0(x)$ is a perfect square, and after performing the birational transformation $y \mapsto y' = \frac{Q(x)}{2y}$ the equation (A.1) for the curve $\widehat{\Sigma}$ turns into the following standard form for a hyperelliptic curve

$$y'^2 = P_2(x),$$

which was the curve considered in [2].

B. The moduli space

In this appendix we begin by reviewing the construction of Krichever and Phong described in [13]. A point in the universal moduli space $\mathcal{U}$ is specified by a smooth Riemann surface $\Sigma_{\mathcal{U}}$ of genus g with N punctures P_α, together with an n_α-jet $[w_\alpha]_{n_\alpha}$ at each puncture. For any positive integer $n_\alpha \in \mathbb{Z}$, an n_α-jet is an equivalence class of local coordinates, w_α, on $\Sigma_{\mathcal{U}}$ defined near the puncture P_α at $w_\alpha = 0$. The equivalence relation is such that two local coordinates w_α and $\tilde{w}_\alpha$ are equivalent if and only if, $\tilde{w}_\alpha = w_\alpha + O(w_\alpha^{n_\alpha+1})$.

In addition, the Riemann surface $\Sigma_{\mathcal{U}}$ is equipped with two meromorphic Abelian integrals E and Q with the following specified behaviour near the punctures P_α,

$$\begin{aligned}
E &= w_\alpha^{-n_\alpha} + c_E + R_\alpha^E \log w_\alpha + O(w_\alpha) \\
dE &= d\left(w_\alpha^{-n_\alpha} + O(w_\alpha)\right) + R_\alpha^E \frac{dw_\alpha}{w_\alpha} \\
Q &= \sum_{k=1}^{m_\alpha} c_{\alpha,k} w_\alpha^{-k} + c_Q + R_\alpha^Q \log w_\alpha + O(w_\alpha) \\
dQ &= d\left(\sum_{k=1}^{m_\alpha} c_{\alpha,k} w_\alpha^{-k} + O(w_\alpha)\right) + R_\alpha^Q \frac{dw_\alpha}{w_\alpha}
\end{aligned} \tag{B.1}$$

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The universal moduli space is parametrised by the data described above,

$$\mathcal{U} = \{\Sigma_{\mathcal{U}}, P_\alpha, [w_\alpha]_{n_\alpha}; E, Q\}\,.$$

Its complex dimension is given as,

$$\dim_{\mathbb{C}} \mathcal{U} = 5g - 3 + 3N + \sum_{\alpha=1}^{N} (m_\alpha + n_\alpha).$$

Krichever and Phong introduce a convenient set of holomorphic coordinates on $\mathcal{U}$. These include the independent residues of the meromorphic differentials dE, dQ and QdE. These are labeled as,

$$R^E_\alpha = \mathrm{Res}_{P_\alpha}\, dE, \quad R^Q_\alpha = \mathrm{Res}_{P_\alpha}\, dQ, \quad T_{\alpha,0} = \mathrm{Res}_{P_\alpha}\, (QdE)\,,$$

for $\alpha = 2, \ldots, N$. They also include the additional residues,

$$T_{\alpha,k} = \mathrm{Res}_{P_\alpha}\, \left(w^k_\alpha Q dE \right),$$

for $k = 1, \ldots, m_\alpha + n_\alpha$ and $\alpha = 1, \ldots, N$. The remaining coordinates are the periods of meromorphic differentials around canonical basis cycles a_i and b_i, for $i = 1, \ldots, g$,

$$\begin{aligned} \tau_{a_i,E} &= \int_{a_i} dE, \quad \tau_{b_i,E} = \int_{b_i} dE, \\ \tau_{a_i,Q} &= \int_{a_i} dQ, \quad \tau_{b_i,Q} = \int_{b_i} dQ, \\ s_i &= \int_{a_i} QdE. \end{aligned}$$

Krichever and Phong prove that the $5g - 3 + 3N + \sum_{\alpha=1}^{N}(m_\alpha + n_\alpha)$ functions,

$$\{R^E_\alpha, R^Q_\alpha, T_{\alpha,0}, T_{\alpha,k}, \tau_{a_i,E}, \tau_{b_i,E}, \tau_{a_i,Q}, \tau_{b_i,Q}, s_i\}$$

have linearly independent differentials and therefore define a local holomorphic coordinate system for $\mathcal{U}$. It follows that the joint level sets of the coordinates,

$$\{R^E_\alpha, R^Q_\alpha, T_{\alpha,0}, T_{\alpha,k}, \tau_{a_i,E}, \tau_{b_i,E}, \tau_{a_i,Q}, \tau_{b_i,Q}\}$$

define a smooth g-dimensional foliation of $\mathcal{U}$ independent of the choices made in defining the coordinates. The remaining variables $\{s_i\}$ are holomorphic coordinates on a leaf of this foliation.

We can now analyse the space of holomorphic data for finite-gap solutions as a special case of the construction described above. In the present case, the curve $\Sigma_{\mathcal{U}}$ will be identified with the spectral curve Σ of genus $g = K - 1$. The Abelian integrals E and Q will be identified with the quasi-momentum $p(P)$ and an, as yet unspecified, function $z(P)$ on Σ. We know that $p(P)$ has only simple poles at $\{(+1)^\pm, (-1)^\pm\}$ and no other singularities. We will choose $z(P)$ to have only simple poles at the points $\{\infty^\pm, 0^\pm\}$ and no other singularities.

Thus we are interested in the $N = 8$ case of the construction described above, where $n_\alpha = 1$, $m_\alpha = 0$ at the first four punctures, and $n_\alpha = 0$, $m_\alpha = 1$ at the second four. In this case the universal moduli space has dimension $5g + 29 = 5K + 24$

To make contact with the construction of Krichever and Phong we must start by assuming nothing about Σ, $p(P)$ and $z(P)$ other than the existence of eight punctures on Σ and the behaviour of $p(P)$ and $z(P)$ near the punctures implied by (B.1) with the identifications $E = p$, $Q = z$ for the specific values of n_α and m_α given above. In particular we do not assume that Σ is hyperelliptic. To properly define the moduli space we must identify each of the constraints which are imposed on Σ, $p(P)$ in the text as a level set condition on the holomorphic coordinates on $\mathcal{U}$. These conditions are,

- The only singularities of $p(P)$ are simple poles at $\{(+1)^\pm, (-1)^\pm\}$ and the singular behaviour near these points is given in (4.9). The residues are absorbed in the definition of the local coordinates w_α at these points. The absence of simple poles for dp implies that $R^E_\alpha = 0$ for all α. This yields seven level set conditions.
- The a- and b-periods of dp are given in (4.6). These implies that $\tau_{a_i,E} = 0$ and $\tau_{b_i,E} = 2\pi n_i$ for $i = 1, \ldots, g$: a total of $2g$ level set conditions.

We will also introduce additional level-set conditions by choosing appropriate properties for the second Abelian integral $z(P)$. These conditions will be sufficiently restrictive to allow us to reconstruct $z(P)$ explicitly. In particular,

- The only singularities of $z(P)$ are simple poles at the points $\{\infty^\pm, 0^\pm\}$. The absence of simple poles in dz implies $R^Q_\alpha = 0$ for all α. Thus we have a further seven level set conditions.
- We will choose the a- and b-periods of dz to vanish implying that $\tau_{a_i,Q} = \tau_{b_i,Q} = 0$. This yields $2g$ level set conditions.
- The behaviour of the differential $QdE = zdp$ near each of the eight singular points yields sixteen constraints on the remaining coordinates which we choose to be as follows,

	$\infty^\pm$	$0^\pm$	$+1^\pm$	$-1^\pm$
$T_{\alpha,0}$	$\frac{4\pi R}{\sqrt{\lambda}}$	$-\frac{4\pi L}{\sqrt{\lambda}}$	0	0
$T_{\alpha,1}$	0	0	-2	$+2$

At this point we have imposed a total of $4g + 29 = 4K + 25$ level set conditions on the $5g + 29 = 5K + 24$ coordinates, thus defining a subspace of the required dimension $g = K - 1$. It remains to show that these conditions uniquely fix the second Abelian integral $z(P)$ and that there are no other constraints on the data. In particular, the only additional constraint placed on the curve in the text was that it take the hyperelliptic form (4.4). For consistency, it is therefore necessary that this condition is not independent but is actually a consequence of the other level-set constraints described above.

The hyperelliptic condition is equivalent to demanding the existence of a well-defined function on Σ with exactly two simple poles and no other singularities. To demonstrate the existence of such a function we use the following facts,

1 The condition $\tau_{a_i,Q} = \tau_{b_i,Q} = 0$, together with $R^Q_\alpha = 0$ implies that dz has vanishing periods and residues on Σ. Therefore the Abelian integral $z(P)$ is actually a well-defined function on Σ.

2 The values of the integers m_α at the punctures P_α imply that $z(P)$ has simple poles at the four points $\{\infty^\pm, 0^\pm\}$ and no other singularities on Σ. Thus $z(P)$ is a function of degree four, deg $z = 4$, which takes each complex value exactly four times on Σ counting multiplicities.

3 The values of the integers n_α at the punctures imply that the meromorphic differential dp has double poles at each of the four points $\{(+1)^\pm, (-1)^\pm\}$. By definition of the local coordinates w_α we have $dp \sim -w_\alpha^{-2}$ near each of these points.

4 The vanishing of the coordinates $T_{\alpha,0}$ at the four points $\{(+1)^\pm, (-1)^\pm\}$ implies that the differential zdp has zero residues at these points. Given **3**, this means that the functions $z(P) - z(P_\alpha)$ must have at least a double zero at each of these points. Equivalently dz vanishes at each of these points.

5 The values $T_{\alpha,1} = -2$ at the points $(+1)^\pm$ and $T_{\alpha,1} = +2$ at the points $(-1)^\pm$ imply that $z(P)$ attains the value $+2$ at the first two points and attains -2 at the second two points.

We are now ready to define the required function as,

$$x(P) = \frac{1}{2}\left(z(P) + \sqrt{z(P)^2 - 4}\right) \tag{B.2}$$

This is not obviously single-valued function on Σ because of the apparent branch points at the zeros of the function $f(P) = z^2(P) - 4$. More precisely, simple zeros of $f(P)$ (or more generally zeros of odd order) give rise to branch-points of $x(P)$. In contrast, double zeros of $f(P)$ do not give rise to branch points. To show that $x(P)$ is a well-defined function on Σ, we will now demonstrate that $f(P)$ has only double zeros. From **2** we know that $z(P)$ has degree four and hence that $f(P)$ has degree eight. Thus $f(P)$ has eight zeros on Σ counting multiplicities. From **5**, $f(P)$ vanishes at each of the four points $\{(+1)^\pm, (-1)^\pm\}$ and from **4** these are at least double zeros. As f has order eight these can be at most double zeros. Thus $f(P)$ has exactly four double zeros at the points $\{(+1)^\pm, (-1)^\pm\}$. It follows that $x(P)$ is well defined on Σ. Inverting (B.2) we obtain,

$$z(P) = x(P) + \frac{1}{x(P)} \tag{B.3}$$

As $x(P)$ is a well-defined function it must have a definite degrees deg x. Hence $x(P)$ attains the values 0 and ∞ exactly deg x times. The relation (B.3) implies that $4 = \text{deg } z = 2\text{deg } x$.

It follows that $x(P)$ is a well-defined function on Σ of degree two with exactly two simple poles at the points $\{\infty^\pm\}$. Thus we deduce that Σ is hyperelliptic and can be put in the form (4.4). As a byproduct the second Abelian integral is then uniquely determined by the formula (B.3). For completeness, one may then check that this formula for $z(P)$ automatically reproduces all the properties of $z(P)$ listed above which are used to define the leaf $\mathcal{L}$.

As discussed in the text, the full moduli space we consider, denoted $\mathcal{M}_{\mathbb{C}}^{(2g)}$, is the Jacobian fibration over the leaf $\mathcal{L}$

$$J(\Sigma) \to \mathcal{M}_{\mathbb{C}}^{(2g)} \to \mathcal{L}. \tag{B.4}$$

We can express the coordinates on $\mathcal{L}$ as periods of the differential,

$$\alpha = \frac{\sqrt{\lambda}}{4\pi} z dp.$$

In particular, it is convenient to use rescaled coordinates,

$$S_i = \frac{1}{2\pi i}\frac{\sqrt{\lambda}}{4\pi} s_i = \frac{1}{2\pi i}\int_{a_i}\alpha. \tag{B.5}$$

The coordinates on the fibre $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_g)$ are defined as the image of the divisor γ under the Abel map,

$$\boldsymbol{\theta} = 2\pi \sum_{j=1}^{g} \int_{\infty^+}^{\gamma_j} \boldsymbol{\omega} + \boldsymbol{\theta}_0 \tag{B.6}$$

where $\boldsymbol{\theta}_0$ is the constant vector defined in (5.43), and $\boldsymbol{\omega}$ is the vector of the basis holomorphic differentials $\{\omega_i\}_{i=1}^g$ on Σ with normalisation $\int_{a_i}\omega_j = \delta_{ij}$.

Following Krichever and Phong we can define a symplectic form ω_{2g} on the moduli space $\mathcal{M}_{\mathbb{C}}^{(2g)}$. For this it is useful to first consider the universal curve bundle $\mathcal{N}$ over the leaf $\mathcal{L}$

$$\Sigma \to \mathcal{N} \to \mathcal{L},$$

whose fibre above every point of the base $\mathcal{L}$ is the corresponding curve Σ. As already mentioned, the $\{s_i\}_{i=1}^g$ form a set of coordinates on the base $\mathcal{L}$, and z may be taken as a coordinate on the fibre, so that δz and $\{\delta s_i\}_{i=1}^g$, where δ denotes the exterior derivative on the total space $\mathcal{N}$, form a basis of differentials at every point of $\mathcal{N}$. In this basis, the total exterior derivative of any function f on $\mathcal{N}$ can be separated as

$$\delta f = \frac{\partial f}{\partial z}\delta z + \sum_{i=1}^{g}(\partial_{s_i} f)\delta s_i \equiv df + \delta^{\mathcal{L}} f,$$

where $\delta^{\mathcal{L}}$ denotes the exterior derivative along the leaf $\mathcal{L}$. Note that $\delta z = dz$ and $\delta s_i = \delta^{\mathcal{L}} s_i$. The differential $p\,dz$ on Σ, as in fact any differential on Σ, can be extended to a differential on $\mathcal{N}$ by setting it to zero along δs_i. Consider now the differential $\delta(p\,dz)$ on $\mathcal{N}$

$$\delta(p\,dz) = \sum_{i=1}^{g} \delta s_i \wedge \partial_{s_i}(p\,dz).$$

The key observation is that, while the differential $p\,dz$ is neither single-valued nor holomorphic on Σ, the ambiguities in its definition as well as its pole parts are constant along the leaf $\mathcal{L}$. Therefore $\partial_{s_i}(p\,dz)$ is holomorphic on Σ and can be expanded in the basis of holomorphic differentials $\{\omega_i\}_{i=1}^g$ of Σ

$$\partial_{s_i}(p\,dz) = \sum_{j=1}^{g} \alpha_{ij}\omega_j, \quad \alpha_{ij} \in \mathbb{C}.$$

The constants α_{ij} can be computed easily

$$\alpha_{ik} = \sum_{j=1}^{g} \alpha_{ij} \int_{a_k} \omega_j = \int_{a_k} \partial_{s_i}(p\,dz) = \partial_{s_i} \int_{a_k} p\,dz = -\partial_{s_i} \int_{a_k} \alpha = -\partial_{s_i} s_k = -\delta_{ik},$$

where in the fourth equality we use the fact that p is the Abelian integral of the normalised differential dp. So in fact we have $\partial_{s_i}(p\,dz) = -\omega_i$. As a result, the differential $\delta(p\,dz)$ takes the simple form

$$\delta(p\,dz) = -\sum_{i=1}^{g} \delta s_i \wedge \omega_i. \tag{B.7}$$

Now since the Jacobian $J(\Sigma)$ can be identified with the symmetric product Σ^g/S_g of g copies of the curve Σ via the Abel map, the bundle $\mathcal{M}_{\mathbb{C}}^{(2g)}$ can be naturally viewed as the symmetric product bundle $\mathcal{N}^g$

$$\Sigma^g/S_g \to \mathcal{N}^g \to \mathcal{L},$$

whose fibre above every point of $\mathcal{L}$ is the g-th symmetric power of the curve Σ. The differential $\delta(p\,dz) = \delta p \wedge dz$ on $\mathcal{N}$ can be used to define a symplecitc form ω_{2g} on $\mathcal{N}^g$ by the following expression, symmetric in the points $\gamma_i \in \Sigma, i = 1, \dots, g$,

$$\omega_{2g} = -\frac{\sqrt{\lambda}}{4\pi i} \sum_{j=1}^{g} \delta p(\gamma_j) \wedge dz(\gamma_j).$$

To see how this also defines a symplectic form on the Jacobian bundle $\mathcal{M}_{\mathbb{C}}^{(2g)}$ however requires a change of variable. Using (B.7) it can first of all be rewritten as

$$\omega_{2g} = \frac{\sqrt{\lambda}}{4\pi i} \sum_{i=1}^{g} \delta s_i \wedge \left(\sum_{i=1}^{g} \omega_i(\gamma_j) \right),$$

or using δ to now denote the exterior derivative on $\mathcal{M}_{\mathbb{C}}^{(2g)}$ we can rewrite ω_{2g} explicitly as a symplectic form on $\mathcal{M}_{\mathbb{C}}^{(2g)}$ with the help of the Abel map (B.6), namely

$$\omega_{2g} = \sum_{i=1}^{g} \delta S_i \wedge \delta\theta_j.$$

C. Variations of $\mathcal{E}$ and $\mathcal{P}$ on the moduli space

In this section we will prove the relations (4.15) which describe the variations of the world-sheet momentum, $\mathcal{P}$ and energy $\mathcal{E}$ along the leaf $\mathcal{L}$ defined in the text.

On a smooth Riemann surface of genus g, we choose a canonical basis of one-cycles $\{a_i, b_i\}_{i=1}^{g}$ and define a pairing of meromorphic differentials Ω_1 and Ω_2 via the formula,

$$(\Omega_1 \bullet \Omega_2) = \sum_{l=1}^{g} \left(\int_{a_l} \Omega_1 \int_{b_l} \Omega_2 - \int_{b_l} \Omega_1 \int_{a_l} \Omega_2 \right). \tag{C.1}$$

Let g_1 be an Abelian integral of Ω_1, i.e. $\Omega_1 = dg_1$. The Riemann bilinear identity states,

$$(\Omega_1 \bullet \Omega_2) = 2\pi i \sum_{\text{poles}} \text{Res}\,[g_1 \Omega_2]\,. \tag{C.2}$$

We will apply this identity on the spectral curve Σ with the identifications $\Omega_1 = dp$ (thus $g_1 = p$) and $\Omega_2 = \delta^{\mathcal{L}}(zdp)$. As in Appendix B and in the text, $\delta^{\mathcal{L}}$ denotes a variation along the leaf $\mathcal{L}$. Using (4.6) for the a- and b-periods of dp and the definitions (4.12) of the moduli S_i, (C.2) becomes,

$$2\pi i \frac{8\pi^2}{\sqrt{\lambda}} \sum_{i=1}^{g} n_i \delta^{\mathcal{L}} S_i = 2\pi i \sum_{\text{poles}} \text{Res}\left[p\, \delta^{\mathcal{L}}(zdp)\right].$$

The non-zero residues of $p\,\delta^{\mathcal{L}}(zdp)$ at the punctures are tabulated below,

	$(+1)^{\pm}$	$(-1)^{\pm}$
Residue	$4\pi^2 \kappa_+ \delta^{\mathcal{L}} \kappa_+$	$-4\pi^2 \kappa_- \delta^{\mathcal{L}} \kappa_-$

Thus we obtain,

$$\sum_{i=1}^{g} n_i \delta^{\mathcal{L}} S_i = -\frac{\sqrt{\lambda}}{2} (\kappa_+ \delta^{\mathcal{L}} \kappa_+ - \kappa_- \delta^{\mathcal{L}} \kappa_-) = -\delta^{\mathcal{L}} \mathcal{P}, \tag{C.3}$$

as required.

The second equality in (4.15) is obtained by a similar application of the Riemann bilinear identity (C.2) with $\Omega_1 = dq$ (thus $g_1 = q$) and $\Omega_2 = \delta^{\mathcal{L}}(zdp)$ as before.

D. Singular parts at $x = \pm 1$

For any matrix $\lambda = n^A \sigma^A \in \mathfrak{su}(2)$ one has $\det \lambda = -\left(n^A n^A\right) = -\frac{1}{2}\text{tr}\,\lambda^2$. So the Virasoro constraint (2.6) may be rewritten as

$$\det j_{\pm} = \kappa_{\pm}^2. \tag{D.1}$$

The eigenvalues of $\frac{1}{2} j_{\pm}$ are determined by $\det(\mu \mathbf{1} - \frac{1}{2} j_{\pm}) = 0$ so since $\text{tr}\, j_{\pm} = 0$ we have $\mu^2 = -\det(\frac{1}{2} j_{\pm}) = -\frac{1}{4} \det j_{\pm} = -\frac{1}{4} \kappa_{\pm}^2$, using the Virasoro constraints (D.1), and thus $\mu = \frac{i}{2} \kappa_{\pm}$ or $\mu = -\frac{i}{2} \kappa_{\pm}$.

In order to determine the singular parts at $x = \pm 1$ consider the auxiliary linear problem

$$\left[\partial_\sigma + \frac{1}{2}\left(\frac{j_+}{1-x} - \frac{j_-}{1+x}\right)\right]\boldsymbol{\psi}(x,\sigma,\tau) = 0, \quad \left[\partial_\tau + \frac{1}{2}\left(\frac{j_+}{1-x} + \frac{j_-}{1+x}\right)\right]\boldsymbol{\psi}(x,\sigma,\tau) = 0. \tag{D.2}$$

We know from (5.4) that $\boldsymbol{\psi}$ takes the following form near $x = \pm 1$

$$\boldsymbol{\psi}(x,\sigma,\tau) = e^{g^\pm(x,\sigma,\tau)}\left(\boldsymbol{\psi}_0^\pm(\sigma,\tau) + \sum_{s=1}^\infty \boldsymbol{\psi}_s^\pm(\sigma,\tau)(1\mp x)^s\right) \quad \text{as } x \to \pm 1,$$

where $g^\pm(x,\sigma,\tau) = \frac{g_1^\pm(\sigma,\tau)}{1\mp x}$ is a singular part at $x = \pm 1$. Consider the point $x = 1$ first. Plugging this expression for $\boldsymbol{\psi}$ back into (D.2) and solving it to lowest order in $(1-x)$ we obtain

$$\left[(\partial_\sigma g_1^+)\mathbf{1} + \frac{1}{2}j_+\right]\boldsymbol{\psi}_0^+(\sigma,\tau) = 0, \quad \left[(\partial_\tau g_1^+)\mathbf{1} + \frac{1}{2}j_+\right]\boldsymbol{\psi}_0^+(\sigma,\tau) = 0.$$

If $\boldsymbol{\psi}_0^+$ is to be non-zero, this means that $-(\partial_\sigma g_1^+) = -(\partial_\tau g_1^+)$ is an eigenvalue of $\frac{1}{2}j_+$ (with corresponding eigenvector $\boldsymbol{\psi}_0^+$). Hence $(\partial_\sigma g_1^+) = (\partial_\tau g_1^+) = \pm\frac{i}{2}\kappa_+$, where the sign is different on both sheets, and in accordance with the implicit choice taken in deriving equation (3.8) one should take the positive eigenvalue on the physical sheet. Doing the same thing for the point $x = -1$ and working to lowest order in $(1+x)$ yields

$$\left[(\partial_\sigma g_1^-)\mathbf{1} - \frac{1}{2}j_-\right]\boldsymbol{\psi}_0^-(\sigma,\tau) = 0, \quad \left[(\partial_\tau g_1^-)\mathbf{1} + \frac{1}{2}j_-\right]\boldsymbol{\psi}_0^-(\sigma,\tau) = 0,$$

so that $(\partial_\sigma g_1^-) = -(\partial_\tau g_1^-) = \pm\frac{i}{2}\kappa_-$, where again the sign is different on both sheets and should be positive on the physical sheet.

We make the choice of normalising the singular part of the eigenvector $\boldsymbol{\psi}(x,\sigma,\tau)$ by the following initial condition on $g^\pm(x,\sigma,\tau)$

$$g^\pm(x,0,0) = 0.$$

This can be achieved simply by normalising the solution $\boldsymbol{\psi}(x,\sigma,\tau)$ to the auxiliary linear problem by a function of $P \in \Sigma$ independent of σ and τ. Integrating up the equations for $g_1^\pm$ with these initial conditions yields the following singular parts at $x = +1$ and $x = -1$ respectively

$$\begin{cases} S_+(x^\pm,\sigma,\tau) = \mp\dfrac{i\kappa_+}{2}\dfrac{\sigma+\tau}{x-1}, \\ S_-(x^\pm,\sigma,\tau) = \mp\dfrac{i\kappa_-}{2}\dfrac{\sigma-\tau}{x+1}. \end{cases} \tag{D.3}$$

E. Motion of the dynamical divisor

The dynamical divisor $\gamma(\sigma,\tau) = \prod_{i=1}^g \gamma_i(\sigma,\tau)$ is the divisor of zeroes of the Baker-Akhiezer function $\varphi(P,\sigma,\tau)$ introduced in section 5.1. To determine its equations of motion on Σ we follow [19] and consider the functions $\frac{\partial_\sigma\varphi}{\varphi}$ and $\frac{\partial_\tau\varphi}{\varphi}$ on Σ. These functions are meromorphic with g poles at $\{\gamma_i(\sigma,\tau)\}_{i=1}^g$ and 4 further poles at the points on Σ above $x = \pm 1$ with the residues

$$\begin{aligned}
&\frac{\partial_\sigma \varphi}{\varphi}(x^\pm) \underset{x\to+1}{\sim} \mp \frac{i\kappa}{2}\frac{1}{x-1} + O(1),\\
&\frac{\partial_\sigma \varphi}{\varphi}(x^\pm) \underset{x\to-1}{\sim} \mp \frac{i\kappa}{2}\frac{1}{x+1} + O(1)\\
&\text{and } \frac{\partial_\tau \varphi}{\varphi} \underset{x\to\pm 1}{\sim} \pm \frac{\partial_\sigma \varphi}{\varphi}
\end{aligned} \tag{E.1}$$

which follow from the behaviour (5.6) of φ at the essential singularities $x = \pm 1$. By the Riemann-Roch theorem the space of functions with $g+4$ prescribed poles and with prescribed residues at four of these poles, as in (E.1), is 1 dimensional. The functions $\frac{\partial_\sigma \varphi}{\varphi}, \frac{\partial_\tau \varphi}{\varphi}$ being meromorphic on the algebraic curve $\Sigma : y^2 = \prod_{i=1}^{2g+2}(x - x_i)$ they can be written as rational functions in the variables x and y.

The most general rational function with poles at $\{\gamma_i(\sigma,\tau)\}_{i=1}^g$ and $x = \pm 1$ is

$$F(x,\sigma,\tau) = \frac{f(x,\sigma,\tau)y + h(x,\sigma,\tau)}{(x-1)(x+1)\tilde{B}(x)}$$

where $\deg f = 1$ and $\deg h = g+2$. First note that it is regular as $x \to \infty$ (since $y \sim x^{g+1}$) so that $x = \infty$ isn't a pole of the function, as required. Now at $x = \pm 1$ there are poles on both sheets with opposite residues (E.1), which imposes the two conditions $h(\pm 1) = 0$ on h yielding

$$h(x,\sigma,\tau) = q(x,\sigma,\tau) - q(+1,\sigma,\tau)\frac{(x+1)\tilde{B}(x)}{2\tilde{B}(1)} - q(-1,\sigma,\tau)\frac{(x-1)\tilde{B}(x)}{-2\tilde{B}(-1)},$$

where $\deg q = g$. The requirement that the function has poles at the points $\gamma_i = (x_{\gamma_i}, y_{\gamma_i})$ but no pole at $(x_{\gamma_i}, -y_{\gamma_i})$ imposes the extra g conditions $q(x_{\gamma_i},\sigma,\tau) = f(x_{\gamma_i})y_{\gamma_i}$ on q so that h now only contains a single undetermined constant. The asymptotic behaviour of the function as $x \to \pm 1$ is

$$F(x,\sigma,\tau) \underset{x\to+1}{\sim} \frac{f(+1)y(+1)}{2\tilde{B}(1)(x-1)}, \quad F(x,\sigma,\tau) \underset{x\to-1}{\sim} \frac{f(-1)y(-1)}{-2\tilde{B}(-1)(x+1)}$$

where $y(x)$ is to be understood as a multivalued function taking different values on both sheets. The remaining linear function f is determined by fixing the residues at the poles $x = \pm 1$. Thus for the function $\frac{\partial_\sigma \varphi}{\varphi}$ we have

$$f_\sigma(x,\sigma,\tau) = -i\kappa\frac{\tilde{B}(1)(x+1)}{2y_{+1}} + i\kappa\frac{\tilde{B}(-1)(x-1)}{-2y_{-1}},$$

and for the function $\frac{\partial_\tau \varphi}{\varphi}$ we have

$$f_\tau(x,\sigma,\tau) = -i\kappa\frac{\tilde{B}(1)(x+1)}{2y_{+1}} - i\kappa\frac{\tilde{B}(-1)(x-1)}{-2y_{-1}},$$

where y_{+1} and y_{-1} are the values of $y(+1)$ and $y(-1)$ on the sheet corresponding to $p(x)$.

To determine the equation of motion for the zeroes of φ we note that since the zeroes are simple zeroes one has $\varphi(x,\sigma,\tau) \sim (x - x_{\gamma_i})\widetilde{\varphi}(x,\sigma,\tau)$, where $\widetilde{\varphi}(x_{\gamma_i}) \neq 0$, which implies the following asymptotic behaviour near the points γ_i

$$\frac{\partial_{\sigma,\tau}\varphi}{\varphi} \underset{P\to\gamma_i}{\sim} -\frac{\partial_{\sigma,\tau}x_{\gamma_i}}{x - x_{\gamma_i}} + O(1)$$

where $P \in \Sigma$ and $x = \pi(P)$. But from above we also have

$$\frac{\partial_{\sigma,\tau}\varphi}{\varphi} \underset{P\to\gamma_i}{\sim} \frac{2f_{\sigma,\tau}(x_{\gamma_i})y_{\gamma_i}}{(x_{\gamma_i}+1)(x_{\gamma_i}-1)\prod_{j\neq i}(x_{\gamma_i} - x_{\gamma_j})}\frac{1}{x - x_{\gamma_i}} + O(1),$$

and comparing the residue of the pole at γ_i in the above two expressions we obtain

$$\begin{cases} \partial_\sigma x_{\gamma_i} = \dfrac{-2f_\sigma(x_{\gamma_i})y_{\gamma_i}}{(x_{\gamma_i}+1)(x_{\gamma_i}-1)\prod_{j\neq i}(x_{\gamma_i} - x_{\gamma_j})} \\ \partial_\tau x_{\gamma_i} = \dfrac{-2f_\tau(x_{\gamma_i})y_{\gamma_i}}{(x_{\gamma_i}+1)(x_{\gamma_i}-1)\prod_{j\neq i}(x_{\gamma_i} - x_{\gamma_j})}. \end{cases} \tag{E.2}$$

Suppose we now continuously deform the solution by bringing the branch points x_{2g+1} and x_{2g+2} together into a single point. Then by the end of the process the curve Σ acquires a singular point at $x_{2g+1} = x_{2g+2}$ and has genus one less, namely $y^2 = (x - x_{2g+1})^2 \prod_{i=1}^{2g}(x - x_i)$. We know that as a result of shrinking the cut $[x_{2g+1}, x_{2g+2}]$ to a singular point, one of the points of the dynamical divisor $\gamma(\sigma,\tau)$, say $\gamma_g(\sigma,\tau)$, should cease to be dynamical. The equations of motion of x_{γ_g} then imply that once the cut is shrunk, $\gamma_g(\sigma,\tau)$ should end up either at one of the remaining $2g$ branch points x_i of the singular curve or at the singular point x_{2g+1} in order that $\partial_{\sigma,\tau}x_{\gamma_g} = 0$. But unless $\gamma_g(\sigma,\tau)$ ends up at x_{2g+1}, the equations of motion of the remaining x_{γ_i} do not take the form (E.2) corresponding to a solution with a curve of genus $g-1$. The conclusion is that one should view singular points $x_i, i = -\infty, \ldots, 0$ as 'trapped points' of the dynamical divisor.

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